Quasi-complemented Para Distributive Latticoid

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Abstract

In this study, we initially establish essential characteristics of minimal elements within para distributive lattices, which are very crucial to introduce quasi-complementation based on these minimal elements. and demonstrate significant properties of this complementation. We derive an equivalent condition for a para distributive lattice become a quasi-complemented in terms of minimal prime filters. We characterize the quasi-complemented para distributive lattice in terms of annihilators and dense elements and finally derive equivalent conditions for it become a Boolean algebra.

Keywords and phrases: Distributive lattice, minimal elements, dense element, ideal, filter, principal ideal, complementation.

MSC 2010 Classifications: Primary: 06D99; Secondary: 06D20.

Introduction

The notion of Boolean Algebra [4] and various applications in computer science and electronics, such as establishing and simplifying electrical circuits and switching circuits, which are applied in the construction of computer chips, resulted from the axiomatisation of Boole's two valued propositional calculus. Other generalisations that can be deduced are those related to lattice theory, such as distributive lattices and Heyting algebras, and ring theory, such as regular rings, p-rings, and biregular rings, among others. An abstraction that integrates these two streams, called almost distributive lattice (**ADL**), was initially introduced by U.M. Swamy and G.C. Rao [7]. For instance, if $(\mathbb{R}, +, ., 0, 1)$ be a commutative regular ring and we define, for any $a, b \in \mathbb{R}$, $a \vee b = a + b + a_0 b$ and $a \wedge b = a_0 b$, where a_0 is the unique idempotent such that $a\mathbb{R} = a_0\mathbb{R}$, then the system $(\mathbb{R}, \vee, \wedge, 0)$ satisfies all the axioms of a distributive lattice, with the exception of the commutativity of $\vee$, the commutativity of $\wedge$, and the right distributivity of join $\vee$ over meet $\wedge$. These imply all of the others, forming a distributive lattice with zero in the structure. Baer-Stone semi groups can also display a similar structure [8]. The exponential development of ADL theory was done by G.C. Rao and U.M. Swamy [[9] [10]], S. Ravi Kumar [[15], [16], [17], [18], [19], [20]], G.N. Rao [[11], [12], [13], [14]], M. Sambasiva Rao [[25], [26], [27], [28], [29]] and S. Ramesh [[21], [22], [23], [24]] as well as others. Over the past fourty years, they have explored the notion filters, of ideals, and several complementations on the structure. Recently Ramesh Bandaru[[1], [2]] introduced an algebraic structure $(2, 2, 1)$ which similar and almost dual structure of ADL but not exact in 2024 and they proved that all the axioms of the algebra are independent each other and it is called Para Distributive Latticoid (abbreviated as **PDL**). This PDL satisfies almost all the properties of distributive lattice except left distributivity of $\vee$ over $\wedge$ out of four distributive properties and both the commutativites of $\vee$ and $\wedge$. The authors studied the class of filters, ideals, prime ideals and derived equivalent conditions for a PDL to become a distributive lattice.

As we know that, distributive lattices and Boolean algebras have many applications in various sciences because of it's smoothness and flexibility of the structure. The properties commutativity of both the meet and join, distributivity and complementation leads to the facility. But PDL structure is very complex and we can not implement and apply properties of PDL freely in diverse fields, because of lack of these three properties in it and the great advantage of this structure is simplify the complex problems in the divers applications of lattice theory with other generalizations of the complementation of Boolean algebra. In this regard we motivated to identify two spacial classes of elements which may not be commutative with respective $\vee$ and $\wedge$ and these are bottom layers of the structure that are set of minimal elements and dense elements which can not separate directly. To define the complementation, need the least and greatest element but this algebraic structure may not contain least element (zero), so we introduce a complementation between the bottom layer and the greatest element instead of least element that is called quasi-complementation on PDL and hence it characterize in terms of the other bottom layer(dense elements). We observe that the existence of this complementation is need not be unique, and in this regard we establish an equivalence condition for this existence to become unique. Finally we derive necessary and sufficient conditions for a PDL to become a Boolean algebra in terms of minimal prime filters, and annihilators by using the quasi-complementation.

1 Preliminaries

Definition 1.1. [1] An algebra $(V, \vee, \wedge, 1)$ of type $(2, 2, 1)$ is called a Para Distributive Latticoid(**Abbreviated as PDL**), if it assures the subsequent axioms:

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(ii) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$

(iii) $(a \vee b) \wedge b = b$

(iv) $(a \vee b) \wedge a = a$

(v) $a \vee (a \wedge b) = a$

(vi) $a \vee 1 = 1$,

for all $a, b, c \in V$.

For any $a, b \in V$, we say that a is less than or equal to b and write $a \leq b$, if $a \wedge b = a$ or equivalently $a \vee b = b$ and it can be easily observed that " $\leq$ " is a partial order on V. The element "1", in the Definition 1.1., is called the greatest element.

Example 1.2. [1] Let V be a non-empty set. Fix some element $b_0 \in V$. Then, for any $a, b \in V$, define $\vee$ and $\wedge$ on V by

$$a \vee b = \begin{cases} a, & \text{if } b \neq b_0 \\ b_0, & \text{if } b = b_0 \end{cases} \quad \text{and} \quad a \wedge b = \begin{cases} b, & \text{if } b \neq b_0 \\ a, & \text{if } b = b_0. \end{cases}$$

Then $(V, \vee, \wedge, b_0)$ is a disconnected PDL with b_0 as its greatest element.

Throughout this paper, we refer to V as a Para Distributive Latticoid with the greatest element 1.

Lemma 1.3. *[1] For any $a, b \in V$, the following holds:*

(i) $a \wedge a = a$

(ii) $a \vee a = a$

(iii) $(a \wedge b) \vee b = b$

(iv) $a \vee (b \wedge a) = a$

(v) $a \wedge (a \vee b) = a$.

Lemma 1.4. *[1] Let $(V, \vee, \wedge, 1)$ be a PDL. Then for any $a, b, c, d \in V$, we have the following:*

(i) $a \wedge 1 = a$

(ii) $1 \wedge a = a$

(iii) $1 \vee a = 1$

(iv) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(v) $a \vee (b \wedge c) = a \vee (c \wedge b)$

(vi) the operation $\vee$ is associative in V

(vii) $d \vee [a \wedge (b \wedge c)] = d \vee [(a \wedge b) \wedge c]$

(viii) $a \vee (b \vee c) = a \vee (c \vee b)$

(ix) $a \vee b = 1 \Leftrightarrow b \vee a = 1$

(x) $a \vee b = 1 \Rightarrow b \wedge a = a \wedge b$.

Lemma 1.5. *[1] For any $a, b \in V$, the following holds:*

(i) $(a \vee b) \vee b = a \vee b$

(ii) $(a \vee b) \vee a = a \vee b$

(iii) $a \vee (a \vee b) = a \vee b$

(iv) $a \wedge (a \wedge b) = a \wedge b$

(v) $(a \wedge b) \wedge b = a \wedge b$

(vi) $b \wedge (a \wedge b) = a \wedge b$.

Theorem 1.6. *[1] Let $(V, \vee, \wedge, 1)$ be a PDL. Then the following are equivalent:*

(i) $(V, \vee, \wedge, 1)$ is a distributive lattice

(ii) The poset $(V, \leq)$ is directed below

(iii) $(a \wedge b) \vee a = a$, for all $a, b \in V$

(iv) The operation $\wedge$ is commutative

(v) The operation $\vee$ is commutative

(vi) The relation $R = \{(a, b) \in V \times V \mid b \vee a = b\}$ is antisymmetric.

Definition 1.7. [1] Let V be a PDL. Then, an element $m \in V$ is said to be a minimal element, if for any $a \in V,\ a \leq m$ implies $\Rightarrow m = a$.

Lemma 1.8. *[1] Let V be a PDL. Then for any $a \in V$, the following are equivalent:*

(i) *m is minimal*

(ii) *$a \wedge m = m$, for all $a \in V$*

(iii) *$a \vee m = a$, for all $a \in V$.*

A non-empty subset F of a PDL V is said to be a filter, if it satisfies $a, b \in F$ implies $a \wedge b \in F$, and $a \in F,\ x \in V$ implies $x \vee a \in F$.

Let S be a non-empty subset of V. Then $[S) = \{x \vee (\wedge_{i=1}^{n} s_i) \mid s_i \in S,\ x \in V,\ 1 \leq i \leq n$ and n is a positive integer$\}$ is the smallest filter of V containing S. If $S = \{a\}$, then we write $[S) = [a)$, the principal filter generated by a. A proper filter(ideal) P of V is said to be a prime filter(ideal) if for any $x, y \in V, x \vee y \in P(x \wedge y \in P) \Rightarrow x \in P$ or $y \in P$.

The collection $F(V)$ of all filters of a PDL V forms a distributive lattice under set inclusion, in which, the glb and lub of any two filters F and G are given by $F \wedge G = F \cap G$ and $F \vee G = \{f \wedge g \mid f \in F$ and $g \in G\}$, respectively and also the set of prime filter $PF(V)$ of a PDL V forms a distributive lattice.

Lemma 1.9. *[1] Let V be a PDL and F is a filter of V. Then, for any $a, b \in V$, we have the following:*

(i) $[a) = \{x \vee a \mid x \in V\}$

(ii) $a \in [b) \Leftrightarrow a = a \vee b$, *for all* $a, b \in V$

(iii) $a \vee b \in F \Leftrightarrow b \vee a \in F$

(iv) $[a \vee b) = [b \vee a) = [a) \wedge [b)$

(v) $[a \wedge b) = [b \wedge a) = [a) \vee [b)$.

Definition 1.10. [2] For any non-empty subset S of a PDL V, write $(S)^* = \{a \in V \mid s \vee a = 1,$ for all $s \in S\}$. Then $(S)^*$ is a filter of V, and is called the annihilator of S in V. If $S = \{s\}$, we write $(s)^*$, for $(\{s\})^*$. A filter F is called an annihilator filter if $F^{**} = F$.

Lemma 1.11. *[2] For any $a, b \in V$,*

(i) $a \leq b \Rightarrow (a)^* \subseteq (b)^*$,

(ii) $(a \vee b)^* = (b \vee a)^*$,

(iii) $(a \wedge b)^* = (b \wedge a)^*$,

(iv) $(a \wedge b)^* = (a)^* \cap (b)^*$,

(v) $(a)^* \vee (b)^* \subseteq (a \vee b)^*$,

(vi) $a \in (x)^* \Rightarrow (x)^{**} \subseteq (a)^*$,

2 Quasi-complemented PDLs

Lemma 2.1. *Let V be a PDL with minimal element m. Then $m \wedge a$ and $a \wedge m$ are minimal, for all $a \in V$.*

Proof. Let m be a minimal element in V and $a \in V$. Then $a \wedge m = m$ and $a \vee m = a$. Now, $(a \wedge m) \vee m = m$. Then $a \wedge m \leq m$. Since m is minimal, $a \wedge m = m$. Hence $a \wedge m$ is minimal.

Therefore $a \wedge m$ is minimal(since m is minimal).
For any $b \in V$,

$$\begin{array}{rclcll} b \leq m \wedge a & \Rightarrow & b \vee (m \wedge a) & = & (m \wedge a) & \\ & \Rightarrow & (b \vee m) \wedge (b \vee a) & = & m \wedge a & (\text{ by Definition } 1.1(i)) \\ & \Rightarrow & b \wedge (b \vee a) & = & m \wedge a & (\text{ since } b \vee m = b) \\ & \Rightarrow & b & = & m \wedge a. & (\text{ Lemma } 1.3(v)) \end{array}$$

Therefore $m \wedge a$ is minimal. ☐

Lemma 2.2. *In a PDL V, $a \wedge b$ is minimal if and only if $b \wedge a$ is minimal, for all $a, b \in V$.*

Proof. Let $a, b \in V$. Suppose $a \wedge b$ is minimal. Let $u \in V$ such that $u \leq b \wedge a$. Now $b \wedge a = u \vee (b \wedge a) = u \vee (a \wedge b)$ (by Lemma 1.4(v)). Since $a \wedge b$ is minimal, $b \wedge a = u$ (by Lemma 1.8(iii)). Therefore $b \wedge a$ is minimal. The converse is trivial. ☐

Theorem 2.3. *Let V be a PDL with minimal elements and M is the set of all minimal elements in V. Then $M \cup \{1\}$ forms a subPDL of V.*

Proof. Let $m_1, m_2 \in M$. Then $m_1 \vee m_2 = m_1$ and $m_1 \wedge m_2 = m_2$. Therefore $m_1 \wedge m_2,\ m_1 \vee m_2 \in M$. Now, $m \vee 1 = 1 \vee m = 1$ and $m \wedge 1 = m = 1 \wedge m$ is minimal, for all $m \in V$. Therefore $M \cup \{1\}$ is subPDL of V. ☐

Theorem 2.4. *The set of minimal elements of a PDL forms an ideal of the PDL.*

Proof. From the above theorem, the set of minimal elements M is closed under $\wedge$ and $\vee$. Let $m \in M$ and $a \in V$. Then $m \wedge a,\ a \wedge m \in M$(by Lemma 2.1). Hence M is an ideal of the PDL V. ☐

Lemma 2.5. *For any element $m \in V$, m is minimal if and only if $[m) = V$.*

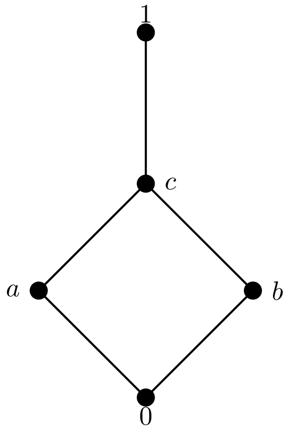
Proof. Let $a, b \in V$. Suppose m is minimal element in V. Then $[m) \subseteq V$. Now, $[a \vee m) = [a) \Rightarrow [a) \cap [m) = [a) \Rightarrow [a) \subseteq [m) \Rightarrow a \in [m)$. Therefore $V \subseteq [m)$. Hence $V = [m)$. Conversely suppose $[m) = V$. If $a < m$, $[m) \subseteq [a) \Rightarrow V = [a) = [m)$. Therefore $a \in [m)$. Then $a = a \vee m$(by Lemma 1.9(ii)). Since $a \leq m$, $m = a \vee m$. Hence $m = a$. Which is a contradiction. Thus m is minimal. ☐

Theorem 2.6. *Let V be a PDL and $a \in V$. Then $[a) = [1)$ if and only if $a = 1$.*

Proof. Suppose $[a) = [1)$. Now, $a \in [1) = [a)$. Then $a = a \vee 1 = 1$(by Lemma 1.9(ii)). Converse is straight forward. ☐

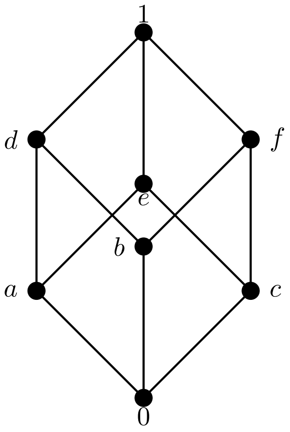
We can observe that the following examples to study independence of prime filters and annihilator filters in PDLs.

Example 2.7. Let $V = \{0, a, b, c, 1\}$ be a PDL whose Hasse-diagram is given below;



Let $F = [a) = \{a, c, 1\}$ is a prime filter of V and $(a)^* = \{1\}$ and $(a)^{**} = \{1\}^* = V$. Then $(a)^{**} \neq [a)$. Therefore $[a)$ is not an annihilator filter in V with respect to a.

Example 2.8. Let $V = \{0, a, b, c, d, e, f, 1\}$ be a PDL whose Hasse-diagram is given below;



Let $F = \{1, d\}$. Then $F^* = \{c, e, f, 1\}$ and $F^{**} = \{d, 1\} = F$. Therefore F is an annihilator filter in V, but not prime. Because let $a, b \in V$, then $a \vee b = d \in F$ but $a \notin F$ and $b \notin F$.

Lemma 2.9. *If every prime filter of a PDL is an annihilator filter, then every prime filter is minimal.*

Proof. Let V be a PDL. Suppose every prime filter of V is an annihilator filter. Let P be a prime filter of V. If P is not minimal, then there exists a minimal prime filter Q of V such that $Q \subset P$. Therefore there exists $a \in P$ such that $a \notin Q$. Since P is prime filter, P is an annihilator filter. Therefore $P = P^{**}$. Now, $a \in P$. Then $a \vee b = 1$, for all $b \in P^*$ and $a \vee b \in Q$. Since Q is prime and $a \notin Q,\ b \in Q$, for all $b \in P^*$. Therefore $P^* \subseteq Q$ and hence $Q^* \subseteq P = P^{**}$. But we have $Q \subsetneq P,\ P^* \subseteq Q^*$. Therefore $P^* \subseteq Q \cap Q^* = [1)$. So that $P^* = [1)$ and $P^{**} = P = ([1))^* = V$. Which is contradiction(since P is proper). Thus P is minimal. □

Definition 2.10. A PDL V is said to be quasi-complemented if for each $a \in V$, there exists $a^{'} \in V$ such that $a \vee a^{'} = 1$ and $a \wedge a^{'}$ is minimal. Hence $a^{'}$ is called quasi-complement of a.

We can observe that the operations $\wedge$ and $\vee$ need not be commutative in PDL V, but $a \vee b = 1$ if and only if $b \vee a = 1$ and $a \wedge b$ is minimal if and only if $b \wedge a$ is minimal, for all $a, b \in V$. Therefore we conclude that a is quasi-complement of b, then b is also a quasi-complement of a.

Example 2.11. Let V be a PDL with minimal elements. Then $M \cup \{1\}$ is a quasi-complemented PDL with same operations on V. Where M is the set of minimal elements in V.

Proof. By Theorem 2.3., $M \cup \{1\}$ is a subPDL of V. Let $m_1 \in M$. Then $m_1 \vee 1 = 1 \vee m_1 = 1$ and $m_1 \wedge 1 = m_1 = 1 \wedge m_1$ is minimal. Therefore $M \cup \{1\}$ is quasi-complemented. ☐

Example 2.12. [1] Let V be a non-empty set. Fix some element $b_0 \in V$. Then for any $a, b \in V$, define $\wedge$ and $\vee$ by

$$a \wedge b = \begin{cases} b, & \text{if } b \neq b_0 \\ a, & \text{if } b = b_0 \end{cases} \quad \text{and} \quad a \vee b = \begin{cases} a, & \text{if } b \neq b_0 \\ b, & \text{if } b = b_0. \end{cases}$$

Let $(V, \wedge, \vee, b_0)$ be disconnected whose operation of $\wedge$ and $\vee$ define in Example 1.2., with greatest element b_0. Moreover V is a quasi-complemented PDL. Because let $a \in V$ and $a \neq b_0$. Then $a \vee b_0 = b_0$. For any $u \in V$, $u \leq a \Rightarrow u \wedge a = u$. Since $a \neq b_0$, $u \wedge a = a$. Therefore $a = u$. Hence $a \wedge b_0 = a$ is minimal. Hence a is quasi-complement of b_0. Thus V is quasi-complemented.

We know that the complement of an element of a distributive lattice(Boolean algebra) is unique if it exists but in PDL need not be unique. In the above example, we observe that the quasi-complement of every non-greatest element($a \neq b_0$) is b_0. Therefore we can state that the quasi-complement of an element in PDL need not be unique.

Theorem 2.13. *Let V be a quasi-complemented PDL and $a, b \in V$. If b is a quasi-complement of a, then $(a)^* = [b)$.*

Proof. Assume that V is a quasi-complemented PDL. For any $a \in V$, there exists $b \in V$ such that $a \vee b = 1$ and $a \wedge b$ is minimal. Then $b \in (a)^*$. Therefore $[b) \subseteq (a)^*$. Let $c \in (a)^*$. Then $c \vee a = 1$. Now, $c = c \vee (a \wedge b) = (c \vee a) \wedge (c \vee b) = 1 \wedge (c \vee b) = c \vee b$. Therefore $c = c \vee b$. By Lemma 1.9(ii)., $c \in [b)$. So that $(a)^* \subseteq [b)$. Hence $(a)^* = [b)$. ☐

Theorem 2.14. *If every element of a quasi-complemented PDL has unique quasi-complement, then it is a Boolean algebra.*

Proof. Let V be a quasi-complemented PDL in which every element has unique quasi-complement. Since $1 \in V$, there exists $1' \in V$ such that $1 \vee 1' = 1 = 1' \vee 1$ and $1' \wedge 1$ is minimal in V. Let m be a minimal element in V. Then $1 \vee m = 1$ and $1 \wedge m = m$ is minimal. Therefore 1 is quasi-complement of m. Since V has unique complement, therefore $m = 1'$. For $a \in V$, there exists $a' \in V$ such that $a \vee a' = 1$ and $a \wedge a'$ is minimal and hence $a \wedge a' = 1'$. Now $(a \wedge a') \vee a = (a' \wedge a) \vee a = a$(by Lemma 1.4(x)). Then $1' = a \wedge a' \leq a$. Therefore $1' \leq a$, for all $a \in V$. Hence V is bounded. Thus V is Boolean algebra. ☐

Theorem 2.15. *For a PDL V with minimal elements, V is quasi-complemented if and only if $PF(V)$ is a Boolean algebra.*

Proof. Suppose V is a quasi-complemented PDL. We know that $PF(V)$ is a bounded distributive lattice with least element $[1)$ and the greatest element $V = [m)$ where m is minimal element in V. Let $a \in V$. Then there exists $b \in V$ such that $a \vee b = 1$ and $a \wedge b$ is minimal. Therefore $[a \vee b) = [a) \wedge [b) = [1)$ and $V = [a \wedge b) = [a) \vee [b)$. Hence $PF(V)$ is Boolean algebra. Conversely suppose $PF(V)$ is a Boolean algebra. Let $a \in V$. Then there exists $b \in V$ such that $[a) \wedge [b) = [1)$ and $[a) \vee [b) = V$. By Lemma 2.5., and Theorem 2.6., $[a \vee b) = [1)$ and $[a \wedge b) = V$. So that $a \vee b = 1$ and $a \wedge b$ is minimal. Hence V is quasi-complemented. ☐

Theorem 2.16. *Let V be a PDL with minimal element m. Then V is quasi-complemented if and only if every prime filter in V is minimal.*

Proof. Let V be a PDL with a minimal element m. Suppose that V is quasi-complemented. Let M be a prime filter in V. If N is a prime filter of V such that $N \subsetneq M$, then there exists $a \in V$ such that $a \in M$ and $a \notin N$. Since V is quasi-complemented, there exists $b \in V$ such that $a \vee b = 1$ and $a \wedge b$ is minimal. Then $a \in N$ or $b \in N$(since $1 \in N$). Therefore $b \in N$ and hence $a \wedge b \in M$. By Lemma 2.5., $V = [a \wedge b) = M$. Which is contradiction. So that M is minimal prime filter of V. Conversely suppose that every prime filter in V is minimal. Let $c \in V$ and c has no quasi-complement in V. Let $N = \{b \in V \mid b \wedge c \text{ is minimal}\}$. Since V has minimal elements$(m), m \wedge c$ is minimal(by Lemma 2.1.). Therefore $m \in N$. So that N is non-empty. Let $a, b \in N$. Then $a \wedge c$ and $b \wedge c$ are minimal in V. Let $d \in V$. Now,

$$\begin{array}{lllll}
d \leq (a \vee b) \wedge c & \Rightarrow & d \vee \{(a \vee b) \wedge c\} & = & (a \vee b) \wedge c \\
 & \Rightarrow & d \vee \{(a \wedge c) \vee (b \wedge c)\} & = & (a \vee b) \wedge c \quad \text{(by Lemma 1.4(iv))} \\
 & \Rightarrow & d \vee (a \wedge c) & = & (a \vee b) \wedge c \quad \text{(since } b \wedge c \text{ is minimal)} \\
 & \Rightarrow & d & = & (a \vee b) \wedge c. \quad \text{(since } a \wedge c \text{ is minimal)}
\end{array}$$

.

Therefore $(a \vee b) \wedge c$ is minimal. So that $(a \vee b) \in N$. Let $e \in V$. Now,

$$\begin{array}{lllll}
d \leq (a \wedge e) \wedge c & \Rightarrow & d \vee \{(a \wedge e) \wedge c\} & = & (a \wedge e) \wedge c \\
 & \Rightarrow & d \vee \{c \wedge (a \wedge e)\} & = & (a \wedge e) \wedge c \quad \text{(by Lemma 1.4(v))} \\
 & \Rightarrow & d \vee \{(c \wedge a) \wedge e\} & = & (a \wedge e) \wedge c \quad \text{(by Lemma 1.4(vii))} \\
 & \Rightarrow & d \vee \{(e \wedge (c \wedge a)\} & = & (a \wedge e) \wedge c \quad \text{(by Lemma 1.4(v))} \\
 & \Rightarrow & d \vee (c \wedge a) & = & (a \wedge e) \wedge c \quad (c \wedge a \text{ is minimal})) \\
 & \Rightarrow & d & = & (a \wedge e) \wedge c. \quad (c \wedge a \text{ is minimal})
\end{array}$$

.

Therefore $(a \wedge e) \wedge c$ is minimal. So that $a \wedge e \in N$.

$$\begin{array}{lllll}
d \leq (e \wedge a) \wedge c & \Rightarrow & d \vee \{(e \wedge a) \wedge c\} & = & (e \wedge a) \wedge c \\
 & \Rightarrow & d \vee \{e \wedge (a \wedge c)\} & = & (e \wedge a) \wedge c \quad \text{(by Lemma 1.4(vii))} \\
 & \Rightarrow & d \vee \{(a \wedge c) \wedge e\} & = & (e \wedge a) \wedge c. \quad \text{(by Lemma 1.4(v))} \\
 & \Rightarrow & d \vee (a \wedge c) & = & (e \wedge a) \wedge c. \quad \text{(since } a \wedge c \text{ is minimal)}
\end{array}$$

Therefore $(e \wedge a) \wedge c$ is a minimal. So that $e \wedge a \in N$. Hence N is an ideal of V. Take an ideal $N_c = N \vee (c]$. If $1 \in N \vee (c]$, then $n \vee (a \wedge c) = (n \vee a) \wedge (n \vee c) = 1$. Therefore $n \vee a = 1 = n \vee c$. Since $n \wedge c$ is minimal, c is a quasi-complement of n. Which is contradiction. So that $1 \notin N_c \vee (c]$. Hence $N_c \cap [1) = \phi$. Which implies that there exists a prime filter P such that $[1) \subseteq P$ and $P \cap N_c = \phi$. Let m be a minimal element in V. If $m \in P \vee [c)$, then $m = p \wedge t$, where $t = t \vee c$. Now $V = [m) = [p \wedge t) = [p) \vee [t) \subseteq [p) \vee [c) = [p \wedge c)$. Then $V = [p \wedge c)$. Therefore $p \wedge c$ is minimal in V(by Lemma 2.5). So that $p \in N \subseteq N_c$. Hence $P \cap N_c \neq \phi$ which is contradiction. Now $P \vee [c)$ is proper filter of V. Then there exists a minimal prime filter M such that $M \subseteq P \vee [c)$. Since $c \notin P,\ M \subsetneqq P$ which is contradiction to our assumption. Thus V is quasi-complemented. □

From the above Theorem, we can directly state the following theorem.

Theorem 2.17. *If every prime filter of a PDL V with minimal elements is an annihilator filter, then V is quasi-complemented.*

3 Characterizations of quasi-complemented PDLs interms of dense elements

Definition 3.1. An element $d \in V$ is said to be dense if V has exactly one element 1 such that $d \vee 1 = 1 \vee d = 1$. Hence $(d)^* = [1) = \{1\}$.

Theorem 3.2. *The set D of dense elements in a PDL V forms an ideal of V whenever V has dense elements.*

Proof. Let $d_1, d_2 \in D$ and $t \in V$. Now, $t \in (d_1 \vee d_2)^*$. Then $t \vee (d_1 \vee d_2) = 1$. Therefore $(t \vee d_1) \vee d_2 = 1$. So that $t \vee d_1 \in (d_2)^* = \{1\}$. Hence $t \vee d_1 = 1$. Since d_1 is dense, $t = 1$. Thus D is closed under $\vee$. Let $a \in V$ and $d_1 \in D$. Now, $t \in (d_1 \wedge a)^*$. Then $t \in (d_1)^* \cap (a)^*$ (by Lemma 1.11(iv)). Therefore $t \in (d_1)^* = \{1\}$. So that $t = 1$. Hence $d_1 \wedge a \in D$. Thus D is an ideal of V. ☐

Lemma 3.3. *Every minimal element of PDL is dense.*

Proof. Let V be a PDL and m is a minimal element in V. For $t \in V$, $t \in (m)^*$. Then $t \vee m = 1 = m \vee t$. Since m is minimal, $t \vee m = t$. Therefore $t = 1$. Hence $(m)^* = \{1\}$. Thus m is dense. ☐

Remarks 3.4. The converse of the above lemma need not be true. We can see the following example.

Example 3.5. Let $V = \{a, b, c, d, 1\}$ be a set with binary operations $\vee$ amd $\wedge$ given in the following tables:

$\vee$	a	1	b	c	d
a	a	1	a	c	c
1	1	1	1	1	1
b	b	1	b	d	d
c	c	1	c	c	c
d	d	1	d	d	d

$\wedge$	a	1	b	c	d
a	a	a	b	a	b
1	a	1	b	c	d
b	a	b	b	a	b
c	a	c	b	c	d
d	a	d	b	c	d

Then $(V, \vee, \wedge, 1)$ is a para distributive latticoid but not distributive. Let $c, d \in V$. Then $(c)^* = (d)^* = \{1\}$. Therefore c and d are dense elements in V, but not minimal. Because $a \wedge c = a \neq c$ and $b \wedge d = b \neq d$. Therefore c and d are not minimal elements in V.

Example 3.6. Let X be a discrete PDL and I is an infinite set. Then the set $\mathcal{S} := \big\{ f \in X^I \mid |f| = \{i \in I \mid f(i) \neq 1\} \text{ is finite } \big\}$ is a PDL with pointwise operations and there are no dense elements. Hence it is not a quasi-complemented PDL. Thus we conclude that every PDL need not be quasi-complemented.

Theorem 3.7. *If V is a quasi-complemented PDL, then every dense element in V is minimal.*

Proof. Suppose V is quasi-complemented. Let $a \in V$ be a dense element. Then $(a)^* = [1)$. For this $a \in V$, there exists $b \in V$ such that $a \vee b = 1$ and $a \wedge b$ is minimal. Therefore $b \in (a)^* = [1)$. So that $b = 1$. Since $a \wedge b$ is minimal, $a \wedge b = a \wedge 1 = a$ is minimal. Thus every dense element in V is minimal. ☐

Lemma 3.8. *Let V be a quasi-complemented PDL and $x \in V$. Then $x \wedge b$ is dense if b is quasi-complement of x.*

Proof. Suppose V is a quasi-complemented PDL and $x \in V$. If $b \in V$ is a quasi-complement of x, then we have $x \vee b = 1$ and $x \wedge b$ is minimal. Let $t \in V$. Now, $t \in (x \wedge b)^*$. Then $t \vee (x \wedge b) = 1$. Since $x \wedge b$ is minimal, $t = 1$. Hence $x \wedge b$ is dense. ☐

Theorem 3.9. *A PDL V is quasi-complemented if and only if it is sectionally quasi-complemented and processes a dense element.*

Proof. Suppose V is quasi-complemented. Then it can directly prove that V is sectionally quasi-complemented. Now, $1 \in V$. Then there exists $a \in V$ such that $1 \vee a = 1$ and $(1 \wedge a)^* = [1)$(since V is quasi-complemented). Then $(a)^* = [1)$ and hence a is dense(by Definition 3.1). Conversely suppose that the given conditions are hold. Let $d \in V$ be a dense and $a \in V$. Then $I = [a \wedge d,\ 1]$

is quasi-complemented and $a \in I$. Hence there exists $b \in I$ such that $a \vee b = 1$ and $(a \wedge b)^* = [1)$ in I. Let $c \in V$ such that $(a \wedge b) \vee c = 1$. Then $(a \vee c) \wedge (b \vee c) = 1$. Therefore $a \vee c = 1 = b \vee c$. Now, $[c \vee (a \wedge d)] \vee (a \wedge b) = c \vee [a \wedge (d \vee b)] = (c \vee a) \wedge (c \vee d \vee b) = 1 \wedge 1 = 1$. Therefore $c \vee (a \wedge d) \in (a \wedge b)^*$ and $c \vee (a \wedge d) \in I$. Since $(a \wedge b)^* = [1)$, $c \vee (a \wedge d) = [1)$. So that $(c \vee a) \wedge (c \vee d) = 1$. Hence $c \vee a = 1 = c \vee d$. Since d is dense, $c = 1$. Therefore $(a \wedge b)^* = [1)$ in V and $a \vee b = 1$. Hence V is quasi-complemented. ☐

Lemma 3.10. *Let V be a PDL and $a \in V$. Then $(a : D) = \{x \in V \mid x \wedge a \in D\}$ is an ideal of V, where D is non-empty set.*

Proof. Let $x, y \in (a : D)$. Then $x \wedge a, y \wedge a \in D$ and $(x \wedge a)^* = \{1\} = (y \wedge a)^*$. Now, $[(x \vee y) \wedge a]^* = [(x \wedge a) \vee (y \wedge a)]^*$. Therefore $(x \wedge a)^* \vee (y \wedge a)^* \subseteq [(x \vee y) \wedge a]^*$(by Lemma 1.11(v)). For $t \in V$,

$$\begin{aligned}
t \in [(x \vee y) \wedge a]^* &\Rightarrow t \vee [(x \vee y) \wedge a] = 1 \\
&\Rightarrow t \vee (x \wedge a) \vee (y \wedge a) = 1 \quad (\text{ by Lemma 1.4(iv) }) \\
&\Rightarrow [t \vee (x \wedge a)] \vee (y \wedge a) = 1 \quad (\text{ by Lemma 1.4(vi) }) \\
&\Rightarrow t \vee (x \wedge a) = 1 \quad (\text{ since } y \wedge a \in D) \\
&\Rightarrow t = 1. \quad (\text{ since } x \wedge a \in D)
\end{aligned}$$

Therefore $(x \vee y) \wedge a \in D$. Hence $x \vee y \in (a : D)$.

For $t, b \in V$,

$$\begin{aligned}
t \in [(x \wedge b) \wedge a]^* &\Rightarrow t \vee [(x \wedge b) \wedge a] = 1 \\
&\Rightarrow [t \vee (x \wedge b)] \wedge (t \vee a) = 1 \quad (\text{ by Definition 1.1(i) }) \\
&\Rightarrow (t \vee x) \wedge (t \vee b) \wedge (t \vee a) = 1 \quad (\text{ by Definition 1.1(i) }) \\
&\Rightarrow t \vee x = t \vee b = t \vee a = 1 \\
&\Rightarrow t \vee (x \wedge a) = 1 \\
&\Rightarrow t = 1. \quad (\text{ since } x \wedge a \in D)
\end{aligned}$$

Therefore $(x \wedge b) \wedge a \in D$. Hence $x \wedge b \in (a : D)$. Thus $(a : D)$ is an ideal of V. ☐

Definition 3.11. Let V be a PDL and $x, y \in V$. Define a relation θ on V by $\theta = \{(x, y) \mid (x)^* = (y)^*\}$ is a congruence relation on V.

Theorem 3.12. *For any quasi-complemented PDL V, $\theta = \{(a, b) \in V \times V \mid (a; D) = (b; D)\}$.*

Proof. Let $(a, b) \in \theta$. Then $(a)^* = (b)^*$. Let $x \in V$. Now, $x \in (a; D) \Leftrightarrow x \wedge a \in D \Leftrightarrow (x \wedge a)^* = \{1\} \Leftrightarrow (x)^* \cap (a)^* = \{1\} \Leftrightarrow (x)^* \cap (b)^* = \{1\} \Leftrightarrow (x \wedge b)^* = \{1\} \Leftrightarrow x \wedge b \in D \Leftrightarrow x \in (b; D)$. Therefore $(a; D) = (b; D)$. Hence $(a, b) \in \{(a, b) \in V \times V \mid (a; D) = (b; D)\}$. Conversely suppose that $a, b \in V$ and $(a; D) = (b; D)\}$. Since V is quasi-complemented, for $a \in V$, there exists $x \in V$ such that $a \vee x = 1$ and $a \wedge x$ is minimal. Since every minimal element is dense, $x \in (a; D) = (b; D)$. Therefore $x \wedge b$ is dense. Let $y \in (b)^*$. Then $y \vee a \in (b)^*$(since $(b)^*$ is filter). Since $a \in (a)^{**}$, $y \vee a \in (a)^{**}$. Then $y \vee a \in (a)^{**} \cap (b)^*$. Since $x \in (a)^*, [x) \subseteq (a)^*$. Therefore $(a)^{**} \subseteq (x)^*$. So that $y \vee a \in (x)^* \cap (b)^* = (x \wedge b)^* = \{1\}$ and $y \in (a)^*$. Hence $(b)^* \subseteq (a)^*$. Similarly we can prove $(a)^* \subseteq (b)^*$. Therefore $(a)^* = (b)^*$. Hence proved the theorem. ☐

Lemma 3.13. *For any PDL V in which every dense element is minimal, V is quasi-complemented if and only if for any $x \in V$, there exists $y \in V$ such that $x \wedge y$ is dense and $x \vee y = 1$.*

Theorem 3.14. *Let F be a filter of V and P is a prime filter in V which containing F. Then P is minimal filter containing F if and only if for any $x \in P$, there exists $y \notin P$ such that $x \vee y \in F$.*

Proof. Suppose that P is a minimal prime filter containing a filter F in V. Then $V - P$ is a prime ideal of V. Let $x \in P$. Now, $I = (V - P) \vee (x]$, where $(x] = \{a \wedge x \mid a \in V\}$. Then I is an ideal of V. If $I \cap F = \phi$. Then there exists a prime filter Q of V such that $F \subseteq Q$ and $I \cap Q = \phi$.

Therefore $Q \subseteq P$. Hence $F \subseteq Q \subseteq P$. Since P is minimal $Q = P$. Which is contradiction(since $x \in P$). Therefore $I \cap F \neq \phi$. Let $t \in I \cap F$. Then $t \in I$ implies $t = y \vee (a \wedge x)$ for some $y \notin P$ and $a \in V$. Now, $t \vee x = [y \vee (a \wedge x)] \vee x = y \vee x \in F$(since $t \in F$ and F is a filter). Then $y \vee x \in F$ and $y \notin P$. Conversely suppose that for any prime filter P of V, and $x \in P$ there exists $y \notin P$ such that $x \vee y \in F$. Let Q be a prime filter of V such that $F \subseteq Q \subseteq P$. If $x \in P$, then there exists $y \notin P$ such that $x \vee y \in F \subseteq Q$. Since Q is prime and $y \notin P$, $x \in Q$. Therefore $P \subseteq Q$. Hence $P = Q$. Thus P is minimal prime filter of V containing F. ☐

Corollary 3.15. *For any prime filter P of V, P is minimal if and only if for any $x \in P$, $(x)^* \nsubseteq P$.*

Proof. Let $F = \{1\}$ be a filter of V. Then $F \subseteq P$ for any prime filter of P of V. Let $x \in P$. By the above theorem there exists $y \notin P$ such that $x \vee y \in F = \{1\}$. Therefore $x \vee y = 1$. So that $y \in (x)^*$. Hence $(x)^* \nsubseteq P$(since $y \notin P$). ☐

Theorem 3.16. *For any PDL V in which every dense element is minimal, V is quasi-complemented if and only if for any filter F of V with $F \cap D = \emptyset$, there exists a prime filter P of V such that $F \subseteq P$ and $P \cap D = \emptyset$.*

Proof. Let V be a PDL with minimal elements and F is a filter of V such that $F \cap D = \emptyset$. Therefore there exists a prime ideal I of V such that $F \cap I = \emptyset$ and $D \subseteq I$. Hence $V - I$ is a prime filter of V containing F and $V - I \cap D = \emptyset$. First we prove that $V - I$ is minimal. Let $x \in V - I$. Since V is quasi-complemented PDL, there exists $x' \in V$ such that $x \wedge x'$ is minimal and $x \vee x' = 1$. If $x' \in V - I$, then $x \wedge x' \in V - I$. Therefore $[x \wedge x') = V = V - I$(since $x \wedge x'$ is minimal). So that $I = \phi$. Which is contradiction. Hence $x' \notin V - I$. Therefore $x' \in I$. Now, $x \vee x' = 1$. Then $x' \in (x)^*$. Since $x' \notin V - I$, $(x)^* \nsubseteq (V - I) \neq \emptyset$. By the above Corollary 3.14., $V - I$ is a minimal prime filter of V containing F. Conversely suppose that for any filter F of V with $F \cap D = \emptyset$, there exists a prime filter P of V such that $F \subseteq P$ and $P \cap D = \emptyset$. Let $x \in V$. Then $[x) \vee (x)^*$ is a filter of V and $[x) \vee (x)^*$ not contained in any minimal prime filter of V. If $[x) \vee (x)^* \subseteq P$ where P is a minimal prime filter of V, then $[1) \vee (1)^* = V \subseteq P$. Hence $P = V$. Which is contradiction. Suppose $[[x) \vee (x)^*] \cap D \neq \emptyset$. Choose $d \in [[x) \vee (x)^*] \cap D$. Then $d = a \wedge b$ where $a \in [x)$, $b \in (x)^*$ and $d \in D$. Therefore $(x)^{**} \subseteq (b)^*$ and $(x)^* \subseteq (a)^*$. So that $x \vee b = 1$ and $(x \wedge b)^* = (x)^* \cap (b)^* \subseteq (a)^* \cap (b)^* = (a \wedge b)^* = (d)^* = \{1\}$. Hence $x \wedge b$ is dense. Thus V is quasi-complemented. ☐

Finally, we define a definition to prove equivalent conditions for a quasi-complemented PDL to become a Boolean algebra. A PDL V is said to be disjunctive if $(a)^* = (b)^*$ implies $a = b$, for all $a, b \in V$.

Theorem 3.17. *For any quasi-complemented PDL V, the following are equivalent;*

(i) V is disjunctive PDL

(ii) V has exactly one dense element

(iii) V is a Boolean algebra.

Proof. Let V be a quasi-complemented PDL.
$(i) \Rightarrow (ii)$: Suppose V is disjunctive. Let d_1, d_2 are two dense elements in V(since every quasi-complemented PDL has dense elements). Then $(d_1)^* = (d_2)^*$. Therefore $d_1 = d_2$(since V is disjunctive). Hence V has exactly one dense element.
$(ii) \Rightarrow (iii)$: Suppose V has exactly one dense element. Let $x \in V$. Then there exists an element $y \in V$ such that $x \wedge y$ is minimal and $x \vee y = 1$. Since every minimal element is dense and by the hypothesis V has unique dense say that the dense element is d'. Therefore $d' = x \wedge x' \leq x \leq 1$. So that the PDL is bounded. Hence it a Boolean algebra.
$(iii) \Rightarrow (i)$: In a Boolean algebra V, $(a)^* = (b)^*$ implies $a = b$, for all $a, b \in V$. Hence V is disjunctive. ☐

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