μ-semi approaches on μ-preopen sets via grill

R. Anbarasan, M. Anitha and D. Saravanakumar

Research Scholar(17231172091001) Affiliated to Manonmaniam Sundaranar University, Rani Anna Goverenment College for Women, India.
Department of Mathematics, PSN College of Engineering and Technology, India.

Department of Mathematics, Rani Anna Goverenment College for Women, India.
Department of Mathematics, St. Joseph University In Tanzania.

Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R And D Institute of Science and Technology, India.

Abstract

In this paper, we introduce a new class of μ-$\mathcal{G}s_p$-open sets in the grill generalized topological space $(X, \mu, \mathcal{G})$. Also, we define (μ-$\mathcal{G}s_p$, ν-pre)-continuous and μ-$\mathcal{G}s_p$-open(closed) functions and study some of their basic properties. In addition, we obtain a (μ-$\mathcal{G}$, ν-$\mathcal{G}'$)s_p-continuous function and analyze the essential theorems through this function.

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1 Introduction and preliminaries

Levine[17] introduced semiopen sets, semi continuity in topological spaces and he identified that the collection of semiopen sets is finer than topology. Biswas[4] analyzed the characterizations of semicontinuous functions and Crossley et al.[9, 10] discussed the concept of semiclosure in the semitopological properties. Mashhour et al.[21] introduced the concepts of preopen set, pre-interior and pre-closure in a topological spaces. Csaszar [11, 12, 13, 14] introduced the concept of generalized topology and he obtained the concepts of μ-open sets, μ-ξ-open sets (where "ξ" stands for α, semi, pre and β). Further he defined the operators μ-interior (resp. μ-ξ-interior) $i_\mu(A)$ (resp. $i_{\xi_\mu}(A) = \cup\{U : U \in \mu$ (resp. $\mu\xi O(X)$ and $U \subseteq A\}$ and μ-closure (resp. μ-ξ-closure) $c_\mu(A)$ (resp. $c_{\xi_\mu}(A) = \cap\{F : X - F \in \mu$ (resp. $\mu\xi O(X)$ and $A \subseteq F\}$, for a subset A of X, A is μ-semiopen (resp. μ-preopen, μ-α-open, μ-β-open) if $A \subseteq c_\mu(i_\mu(A))$ (resp. $A \subseteq i_\mu(c_\mu(A))$, $A \subseteq i_\mu(c_\mu(i_\mu(A)))$, $A \subseteq c_\mu(i_\mu(c_\mu(A))))$ and initiated generalized continuity such as (μ, ν)-continuous (resp. (μ-ξ, ν)-continuous, (μ, ν)-ξ-continuous) if the inverse image of every ν-open (resp. ν-open, ν-ξ-open) set is μ-open (resp. μ-ξ-open, μ-ξ-open) and studied their essential relationships between the functions through the notions of μ-open sets and μ-ξ-open sets in generalized topological spaces. Saravanakumar et al.[26, 28, 29, 30, 33] initiated $\tilde{\mu}$-open sets $\tilde{\mu}$-separation axioms, operation generalized sets and generalized topological continuities etc.

Choquet[8] introduced the concept of grill on a topological space and the idea of grills has shown to be a essential tool for studying some topological concepts. A non-null collection $\mathcal{G}$ of subsets of a topological space (X, τ) is called a grill on X if (i) $\emptyset \notin \mathcal{G}$, (ii) $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$, (iii) $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$. A triple $(X, \tau, \mathcal{G})$ is called a grill topological space.

Modak[22] extended the concept of grill ideas in generalized topological spaces and he defined the mappings Φ_μ and c^{Φ_μ} and introduced a Φ_μ-generalized structure with respect to μ and grill in generalized topological spaces. For any point $x \in X$, $\mu(x)$ denotes the collection of all μ-open neighborhoods of x. A mapping $\Phi_\mu : P(X) \to P(X)$ is defined by $\Phi_\mu(A) = \{x \in X : A \cap U \in \mathcal{G}$ for all $U \in \mu(x)\}$ for all $A \in P(X)$. A mapping $c^{\Phi_\mu} : P(X) \to P(X)$ is defined by $c^{\Phi_\mu}(A) = A \cup \Phi_\mu(A)$ for all $A \in P(X)$. For $A, B \subseteq X$, the mapping c^{Φ_μ} satisfies (i) $c^{\Phi_\mu}(\emptyset) = \emptyset$ and $c^{\Phi_\mu}(X) = X$, (ii) if $A \subseteq B$, then $c^{\Phi_\mu}(A) \subseteq c^{\Phi_\mu}(B)$, (iii) $c^{\Phi_\mu}(c^{\Phi_\mu}(A)) = c^{\Phi_\mu}(A)$ and (iv) $c^{\Phi_\mu}(A) \cup c^{\Phi_\mu}(B) \subseteq c^{\Phi_\mu}(A \cup B)$.

Corresponding to a grill $\mathcal{G}$ on a generalized topological space (X,μ), there exists a Φ_μ-generalized structure on X given by $\mu^\Phi(\mu;\mathcal{G}) = \{U \subseteq X : c^{\Phi_\mu}(X-U) = X-U\}$, where for any $A \subseteq X$, $c^{\Phi_\mu}(A) = A \cup \Phi_\mu(A)$.

Saravanakumar et al.[27] discussed the concept of decomposition of continuity in grill generalized topological spaces and defined some grill open sets. A subset A in X is said to be (i) Φ_μ-open if $A \subseteq i_\mu(\Phi_\mu(A))$, (ii) μ-$\mathcal{G}$-α-open if $A \subseteq i_\mu(c^{\Phi_\mu}(i_\mu(A)))$, (iii) μ-$\mathcal{G}$-preopen if $A \subseteq i_\mu(c^{\Phi_\mu}(A))$, (iv) μ-$\mathcal{G}$-semiopen if $A \subseteq c^{\Phi_\mu}(i_\mu(A))$, (v) μ-$\mathcal{G}$-β-open if $A \subseteq c_\mu(i_\mu(c^{\Phi_\mu}(A)))$. A subset A of X is called Φ_μ-closed (resp. μ-$\mathcal{G}$-α-closed, μ-$\mathcal{G}$-preclosed, μ-$\mathcal{G}$-semiclosed, μ-$\mathcal{G}$-β-closed) if its complement $X-A$ is Φ_μ-open (resp. μ-$\mathcal{G}$-α-open, μ-$\mathcal{G}$-preopen, μ-$\mathcal{G}$-semiopen, μ-$\mathcal{G}$-β-open). The family of all Φ_μ-open (resp. μ-$\mathcal{G}$-α-open, μ-$\mathcal{G}$-preopen, μ-$\mathcal{G}$-semiopen, μ-$\mathcal{G}$-β-open) sets is denoted by $\Phi_\mu O(X)$ (resp. μ-$\mathcal{G}\alpha O(X)$, μ-$\mathcal{G}PO(X)$, μ-μ-$\mathcal{G}SO(X)$, μ-$\mathcal{G}\beta O(X)$). A function $f:(X,\mu,\mathcal{G}) \to (Y,\nu)$ is said to be (μ-$\mathcal{G}$-semi, ν)-continuous if $f^{-1}(V) \in \mu$-$\mathcal{G}SO(X)$ for each $V \in \nu$.

In this paper, we define the concept of μ-$\mathcal{G}s_p$-open set in a grill generalized topological space $(X,\mu,\mathcal{G})$. Also, we introduce μ-$\mathcal{G}s_p$-interior and μ-$\mathcal{G}s_p$-closure and we study some of their basic properties. Further, we define (μ-$\mathcal{G}s_p$, ν-pre)-continuous, (μ-pre, ν-$\mathcal{G}s_p$)-open (resp. closed) and (μ-$\mathcal{G}S^*$, ν-pre)-continuous functions in a grill generalized topological space $(X,\mu,\mathcal{G})$ and we investigate some of their fundamental properties. Moreover, we introduce a (μ-$\mathcal{G}$, ν-$\mathcal{G}'$)s_p-continuous function and we show that every (μ-$\mathcal{G}$, ν-$\mathcal{G}'$)s_p-continuous function is (μ-$\mathcal{G}s_p$, ν-pre)-continuous, but the converse need not to be true.

Proposition 1.1.[22] Let $(X,\mu,\mathcal{G})$ be a grill generalized topological space. Then for all $A,B \subseteq X$:
(i) $A \subseteq B$ implies that $\Phi_\mu(A) \subseteq \Phi_\mu(B)$,
(ii) $\Phi_\mu(A \cup B) = \Phi_\mu(A) \cup \Phi_\mu(B)$,
(iii) $\Phi_\mu(\Phi_\mu(A)) \subseteq \Phi_\mu(A) = \mathrm{cl}(\Phi_\mu(A)) \subseteq \mathrm{cl}(A)$.

Theorem 1.2.[22] Let $(X,\mu,\mathcal{G})$ be a grill generalized topological space. Then (i) $\Phi_\mu(\emptyset) = \emptyset$,
(ii) for $A,B \subseteq X$ and $A \subseteq B$, $\Phi_\mu(A) \subseteq \Phi_\mu(B)$,
(iii) $\Phi_\mu(A) \subseteq c_\mu(A)$,
(iv) $\Phi_\mu(\Phi_\mu(A)) \subseteq c_\mu(A)$,
(v) $\Phi_\mu(A)$ is a μ-closed set,
(vi) $\Phi_\mu(\Phi_\mu(A)) \subseteq \Phi_\mu(A)$,
(vii) for $\mathcal{G} \subseteq \mathcal{G}'$ implies $\Phi_\mu(A_{(\mathcal{G}',\mu)} \supseteq \Phi_\mu(A_{(\mathcal{G},\mu)}$,
(viii) for $U \in \mu$, $U \cap \Phi_\mu(U \cap A) \subseteq U \cap \Phi_\mu(A)$,
(ix) for $G \notin \mathcal{G}$, $\Phi_\mu(A-G) = \Phi_\mu(A)) = \Phi_\mu(A \cup G))$.

2 μ-$\mathcal{G}s_p$-open sets

Definition 2.1. Let $(X,\mu,\mathcal{G})$ be a grill generalized topological space and let A be a subset of X. Then A is said to be μ-$\mathcal{G}s_p$-open if and only if there exists a set $U \in \mu PO(X)$ such that $U \subseteq A \subseteq c^{\Phi_\mu}(U)$. A set A of X is μ-$\mathcal{G}s_p$-closed if its complement $X-A$ is μ-$\mathcal{G}s_p$-open. The family of all μ-$\mathcal{G}s_p$-open (resp. μ-$\mathcal{G}s_p$-closed) sets is denoted by μ-$\mathcal{G}s_pO(X)$ (resp. μ-$\mathcal{G}s_pC(X)$).

Example 2.1. Let $X = \{a,b,c,d\}$, $\mu = \{\emptyset, X, \{b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}\}$ and $\mathcal{G} = P(X) - \{\emptyset, \{c\}\}$. Then μ-$\mathcal{G}s_pO(X) = \{\emptyset, X, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$.

Theorem 2.1. Let $(X,\mu,\mathcal{G})$ be a grill generalized topological space and $A \subseteq X$. Then $A \in \mu$-$\mathcal{G}s_pO(X)$ if and only if $A \subseteq c^{\Phi_\mu}(i_{p_\mu}(A))$.

Proof. If $A \in \mu$-$\mathcal{G}s_pO(X)$, then there exists a $U \in \mu PO(X)$ such that $U \subseteq A \subseteq c^{\Phi_\mu}(U)$. But $U \subseteq A$ implies that $U \subseteq i_{p_\mu}(A)$. Hence $c^{\Phi_\mu}(U) \subseteq c^{\Phi_\mu}(i_{p_\mu}(A))$. Therefore $A \subseteq c^{\Phi_\mu}(i_{p_\mu}(A))$. Conversely, let $A \subseteq c^{\Phi_\mu}(i_{p_\mu}(A))$. To prove that $A \in \mu$-$\mathcal{G}s_pO(X)$, take $U = i_{p_\mu}(A)$, then $U \subseteq A \subseteq c^{\Phi_\mu}(U)$. Hence $A \in \mu$-$\mathcal{G}s_pO(X)$.

Corollary 2.1. If $A \subseteq X$, then $A \in \mu$-$\mathcal{G}s_pO(X)$ if and only if $c^{\Phi_\mu}(A) = c^{\Phi_\mu}(i_{p_\mu}(A))$.

Proof. Let $A \in \mu\text{-}\mathcal{G}s_pO(X)$. Then as c^{Φ_μ} is monotonic and idempotent, $c^{\Phi_\mu}(A) \subseteq c^{\Phi_\mu}(c^{\Phi_\mu}(i_{p_\mu}(A))) = c^{\Phi_\mu}(i_{p_\mu}(A)) \subseteq c^{\Phi_\mu}(A)$ implies that $c^{\Phi_\mu}(A) = c^{\Phi_\mu}(i_{p_\mu}(A))$. The converse is trivial.

Corollary 2.2. If $A \subseteq X$, then $c^{\Phi_\mu}(i_{p_\mu}(A)) \in \mu\text{-}\mathcal{G}s_pO(X)$.

Proof. Clearly $c^{\Phi_\mu}(i_{p_\mu}(A)) = c^{\Phi_\mu}(i_{p_\mu}(i_{p_\mu}(A)) \subseteq c^{\Phi_\mu}(i_{p_\mu}(c^{\Phi_\mu}(i_{p_\mu}(A))))$. Then by Theorem 2.1, $c^{\Phi_\mu}(i_{p_\mu}(A)) \in \mu\text{-}\mathcal{G}s_pO(X)$.

Theorem 2.2. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space. If $A \in \mu\text{-}\mathcal{G}s_pO(X)$ and $B \subseteq X$ such that $A \subseteq B \subseteq c^{\Phi_\mu}(i_{p_\mu}(A))$, then $B \in \mu\text{-}\mathcal{G}s_pO(X)$.

Proof. Given $A \in \mu\text{-}\mathcal{G}s_pO(X)$. Then by Theorem 2.1, $A \subseteq c^{\Phi_\mu}(i_{p_\mu}(A))$. But $A \subseteq B$ implies that $i_{p_\mu}(A) \subseteq i_{p_\mu}(B)$, hence $c^{\Phi_\mu}(i_{p_\mu}(A)) \subseteq c^{\Phi_\mu}(i_{p_\mu}(B))$. Therefore $B \subseteq c^{\Phi_\mu}(i_{p_\mu}(A)) \subseteq c^{\Phi_\mu}(i_{p_\mu}(B))$. Hence by Theorem 2.1, $B \in \mu\text{-}\mathcal{G}s_pO(X)$.

Corollary 2.3. If $A \in \mu\text{-}\mathcal{G}s_pO(X)$ and $B \subseteq X$ such that $A \subseteq B \subseteq c^{\Phi_\mu}(A)$, then $B \in \mu\text{-}\mathcal{G}s_pO(X)$.

Proof. Follows from the Theorem 2.2 and Corollary 2.1.

Proposition 2.1. If $U \in \mu PO(X)$, then $U \in \mu\text{-}\mathcal{G}s_pO(X)$.

Proof. Let $U \in \mu PO(X)$, it implies that $U = i_{p_\mu}(A) \subseteq c^{\Phi_\mu}(U)$. Hence $U \in \mu\text{-}\mathcal{G}s_pO(X)$.

Note that the converse of the above Proposition need not be true. By Example 2.1, we have that $\mu PO(X) = \{\emptyset, X, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}\}$. Then $\{b,d\} \in \mu\text{-}\mathcal{G}s_pO(X)$, but $\{b,d\} \notin \mu PO(X)$.

Theorem 2.3. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space. If $A \in \mu\text{-}\mathcal{G}SO(X)$, then $A \in \mu\text{-}\mathcal{G}s_pO(X)$.

Proof. Given $A \in \mu\text{-}\mathcal{G}SO(X)$. Then $A \subseteq c^{\Phi_\mu}(i_\mu(A))$. Since $i_\mu(A) \subseteq i_{p_\mu}(A)$, we have that $c^{\Phi_\mu}(i_\mu(A)) \subseteq c^{\Phi_\mu}(i_{p_\mu}(A))$ (by Theorem 3.2[22]). Hence $A \subseteq c^{\Phi_\mu}(i_{p_\mu}(A))$ and thus $A \in \mu\text{-}\mathcal{G}s_pO(X)$.

Note that the converse of the above theorem need not be true. By Example 2.1, we have that $\mu\text{-}\mathcal{G}SO(X) = \{\emptyset, X, \{b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$. Then $\{a,b\} \in \mu\text{-}\mathcal{G}s_pO(X)$, but $\{a,b\} \notin \mu\text{-}\mathcal{G}SO(X)$.

Theorem 2.4. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space. If $\mu PO(X) = \mu$, then $\mu\text{-}\mathcal{G}s_pO(X) = \mu\text{-}\mathcal{G}SO(X)$.

Proof. By Theorem 2.3, we have that $\mu\text{-}\mathcal{G}SO(X) \subseteq \mu\text{-}\mathcal{G}s_pO(X)$. Let $A \in \mu\text{-}\mathcal{G}s_pO(X)$. Then by Theorem 2.1, $A \subseteq c^{\Phi_\mu}(i_{p_\mu}(A))$. Since $\mu PO(X) = \mu$, we have that $i_{p_\mu}(A) = i_\mu(A)$ implies that $A \subseteq c^{\Phi_\mu}(i_{p_\mu}(A)) = c^{\Phi_\mu}(i_\mu(A))$ and hence $A \in \mu\text{-}\mathcal{G}SO(X)$. Thus $\mu\text{-}\mathcal{G}s_pO(X) \subseteq \mu\text{-}\mathcal{G}SO(X)$.

Theorem 2.5. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space. Then the following conditions are hold good: (i) for each $\alpha \in J$, $A_\alpha \in \mu\text{-}\mathcal{G}s_pO(X)$, then $\bigcup_{\alpha \in J} A_\alpha \in \mu\text{-}\mathcal{G}s_pO(X)$,
(ii) if $A \in \mu\text{-}\mathcal{G}s_pO(X)$ and $U \in \mu PO(X)$, then $A \cap U \in \mu\text{-}\mathcal{G}s_pO(X)$.

Proof. (i) Suppose $A_\alpha \in \mu\text{-}\mathcal{G}s_pO(X)$, for each $\alpha \in J$. Then $A_\alpha \subseteq c^{\Phi_\mu}(i_{p_\mu}(A_\alpha))$, for each $\alpha \in J$, implies that $\bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in J} c^{\Phi_\mu}(i_{p_\mu}(A_\alpha)) \subseteq c^{\Phi_\mu}(i_{p_\mu}(\bigcup_{\alpha \in J} A_\alpha))$. Therefore $\bigcup_{\alpha \in J} A_\alpha \in \mu\text{-}\mathcal{G}s_pO(X)$.

(ii) Let $A \in \mu\text{-}\mathcal{G}s_pO(X)$ and $U \in \mu PO(X)$. Then $A \subseteq c^{\Phi_\mu}(i_{p_\mu}(A))$. Now, $A \cap U \subseteq c^{\Phi_\mu}(i_{p_\mu}(A)) \cap U = (i_{p_\mu}(A) \cup \Phi_\mu(i_{p_\mu}(A))) \cap U = (i_{p_\mu}(A) \cap U) \cup (\Phi_\mu(i_{p_\mu}(A)) \cap U) \subseteq i_{p_\mu}(A \cap U) \cup \Phi_\mu(i_{p_\mu}(A) \cap U)$ (by Theorem 1.2) $= i_{p_\mu}(A \cap U) \cup \Phi_\mu(i_{p_\mu}(A \cap U)) = c^{\Phi_\mu}(i_{p_\mu}(A \cap U))$. Therefore $A \cap U \in \mu\text{-}\mathcal{G}s_pO(X)$.

Remark 2.1. The following example shows that if $A, B \in \mu$-$\mathcal{G}s_pO(X)$, then $A \cap B \notin \mu$-$\mathcal{G}s_pO(X)$.

From Example 2.1, take $A = \{a, b\}$, $B = \{a, c\}$. Then $A, B \in \mu$-$\mathcal{G}s_pO(X)$, but $A \cap B = \{a\} \notin \mu$-$\mathcal{G}s_pO(X)$.

Theorem 2.6. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space and $A \subseteq X$. If $A \in \mu$-$\mathcal{G}s_pC(X)$, then $i_{p_\mu}(c^{\Phi_\mu}(A)) \subseteq A$.

Proof. Suppose $A \in \mu$-$\mathcal{G}s_pC(X)$. Then $X - A \in \mu$-$\mathcal{G}s_pO(X)$ and hence $X - A \subseteq c^{\Phi_\mu}(i_{p_\mu}(X - A)) \subseteq c_{p_\mu}(i_{p_\mu}(X - A)) = X - i_{p_\mu}(c_{p_\mu}(A)) \subseteq X - i_{p_\mu}c^{\Phi_\mu}(A))$, implies that $i_{p_\mu}(c^{\Phi_\mu}(A)) \subseteq A$.

Remark 2.2. For $A \subseteq X$, the following example shows that (i) if $i_{p_\mu}(c^{\Phi_\mu}(A)) \subseteq A$, then $A \notin \mu$-$\mathcal{G}s_pC(X)$; (ii) $i_{p_\mu}(c^{\Phi_\mu}(A)) \notin \mu$-$\mathcal{G}s_pC(X)$.

(i) Take $A = \{b, c, d\}$ in Example 2.1, then $i_{p_\mu}(c^{\Phi_\mu}(\{b, c, d\})) = \{b, c, d\} \subseteq \{b, c, d\}$. Therefore $i_{p_\mu}(c^{\Phi_\mu}(A)) \subseteq A$, but $A \notin \mu$-$\mathcal{G}s_pC(X)$.

(ii) From (i), $i_{p_\mu}(c^{\Phi_\mu}(\{b, c, d\})) = \{b, c, d\} \notin \mu$-$\mathcal{G}s_pC(X)$. Thus $i_{p_\mu}(c^{\Phi_\mu}(A)) \notin \mu$-$\mathcal{G}s_pC(X)$.

Theorem 2.7. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space and $A \subseteq X$ such that $X - i_{p_\mu}(c^{\Phi_\mu}(A)) = c^{\Phi_\mu}(i_{p_\mu}(X - A))$. Then the following conditions are hold good:
(i) $A \in \mu$-$\mathcal{G}s_pC(X)$ if and only if $i_{p_\mu}(c^{\Phi_\mu}(A)) \subseteq A$,
(ii) $i_{p_\mu}(c^{\Phi_\mu}(A)) \in \mu$-$\mathcal{G}s_pC(X)$.

Proof. (i) Necessary part is proved by Theorem 2.6. Conversely, suppose that $i_{p_\mu}(c^{\Phi_\mu}(A)) \subseteq A$. Then $X - A \subseteq X - i_{p_\mu}(c^{\Phi_\mu}(A)) = c^{\Phi_\mu}(i_{p_\mu}(X - A))$, implies that $X - A \in \mu$-$\mathcal{G}s_pO(X)$. Hence $A \in \mu$-$\mathcal{G}s_pC(X)$.

(ii) Follows from (i).

Theorem 2.8. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space. If $A_\alpha \in \mu$-$\mathcal{G}s_pC(X)$ for each $\alpha \in J$, then $\bigcap_{\alpha \in J} A_\alpha \in \mu$-$\mathcal{G}s_pC(X)$.

Proof. Let $A_\alpha \in \mu$-$\mathcal{G}s_pC(X)$. Then $X - A_\alpha \in \mu$-$\mathcal{G}s_pO(X)$. By Theorem 2.5(i), $\bigcup_{\alpha \in J}(X - A_\alpha) \in \mu$-$\mathcal{G}s_pO(X)$. This implies that $\bigcup_{\alpha \in J}(X - A_\alpha) = X - \bigcap_{\alpha \in J} A_\alpha \in \mu$-$\mathcal{G}s_pO(X)$ and hence $\bigcap_{\alpha \in J} A_\alpha \in \mu$-$\mathcal{G}s_pC(X)$.

Definition 2.2. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space and $A \subseteq X$. Then
(i) μ-$\mathcal{G}s_p$-interior of A is defined as union of all μ-$\mathcal{G}s_p$-open sets contained in A. Thus μ-$\mathcal{G}s_p\text{int}(A) = \cup\{U : U \in \mu\text{-}\mathcal{G}s_pO(X) \text{ and } U \subseteq A\}$,
(ii) μ-$\mathcal{G}s_p$-closure of A is defined as intersection of all μ-$\mathcal{G}s_p$-closed sets containing A. Thus μ-$\mathcal{G}s_p\text{cl}(A) = \cap\{F : X - F \in \mu\text{-}\mathcal{G}s_pO(X) \text{ and } A \subseteq F\}$.

Theorem 2.9. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space and $A \subseteq X$. Then the following conditions are hold good:
(i) μ-$\mathcal{G}s_p\text{int}(A)$ is a μ-$\mathcal{G}s_p$-open set contained in A,
(ii) μ-$\mathcal{G}s_p\text{cl}(A)$ is a μ-$\mathcal{G}s_p$-closed set containing A,
(iii) A is μ-$\mathcal{G}s_p$-closed if and only if μ-$\mathcal{G}s_p\text{cl}(A) = A$,
(iv) A is μ-$\mathcal{G}s_p$-open if and only if μ-$\mathcal{G}s_p\text{int}(A) = A$,
(v) μ-$\mathcal{G}s_p\text{int}(\mu\text{-}\mathcal{G}s_p\text{int}(A)) = \mu\text{-}\mathcal{G}s_p\text{int}(A)$,
(vi) μ-$\mathcal{G}s_p\text{cl}(\mu\text{-}\mathcal{G}s_p\text{cl}(A)) = \mu\text{-}\mathcal{G}s_p\text{cl}(A)$,
(vii) μ-$\mathcal{G}s_p\text{int}(A) = X - \mu\text{-}\mathcal{G}s_p\text{cl}(X - A)$,
(viii) μ-$\mathcal{G}s_p\text{cl}(A) = X - \mu\text{-}\mathcal{G}s_p\text{int}(X - A)$.

Proof. (i) Follows from the Definition 2.2(i) and Theorem 2.5(i)
(ii) Follows from the Definition 2.2(ii) and Theorem 2.8.
(iii) Follows from the condition (ii) and Definition 2.2(ii).

(iv) Follows from the condition (i) and Definition 2.2(i).
(v) Follows from the conditions (i) and (iv).
(vi) Follows from the conditions (ii) and (iii).
(vii) and (viii) Follows from the Definitions 2.1 and 2.2(i),(ii).

Theorem 2.10. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space and $A, B \subseteq X$. Then the following conditions are hold good:
(i) if $A \subseteq B$, then $\mu\text{-}\mathcal{G}s_p\text{int}(A) \subseteq \mu\text{-}\mathcal{G}s_p\text{int}(B)$,
(ii) if $A \subseteq B$, then $\mu\text{-}\mathcal{G}s_p\text{cl}(A) \subseteq \mu\text{-}\mathcal{G}s_p\text{cl}(B)$,
(iii) $\mu\text{-}\mathcal{G}s_p\text{int}(A \cup B) \supseteq \mu\text{-}\mathcal{G}s_p\text{int}(A) \cup \mu\text{-}\mathcal{G}s_p\text{int}(B)$,
(iv) $\mu\text{-}\mathcal{G}s_p\text{cl}(A \cap B) \subseteq \mu\text{-}\mathcal{G}s_p\text{cl}(A) \cap \mu\text{-}\mathcal{G}s_p\text{cl}(B)$,
(v) $\mu\text{-}\mathcal{G}s_p\text{int}(A \cap B) \subseteq \mu\text{-}\mathcal{G}s_p\text{int}(A) \cap \mu\text{-}\mathcal{G}s_p\text{int}(B)$,
(vi) $\mu\text{-}\mathcal{G}s_p\text{cl}(A \cup B) \supseteq \mu\text{-}\mathcal{G}s_p\text{cl}(A) \cup \mu\text{-}\mathcal{G}s_p\text{cl}(B)$.

Proof. (i) and (ii) Follows from the Definitions 2.2(i) and 2.2(ii) respectively.
(iii) and (iv) Follows from the condition (i), Theorem 2.5(i) and the condition (ii), Theorem 2.8 respectively.
(v) and (vi) Follows from the conditions (i) and (ii) respectively.

Theorem 2.11. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space and $A \subseteq X$. Then the following conditions are hold good:
(i) if $c^{\Phi_\mu}(i_{p_\mu}(A)) \subseteq A$, then $c^{\Phi_\mu}(i_{p_\mu}(A)) \subseteq \mu\text{-}\mathcal{G}s_p\text{int}(A)$,
(ii) if $A \subseteq X$ and $X - i_{p_\mu}(c^{\Phi_\mu}(A)) = c^{\Phi_\mu}(i_{p_\mu}(X - A))$, then $\mu\text{-}\mathcal{G}s_p\text{cl}(A) \subseteq i_{p_\mu}(c^{\Phi_\mu}(A))$.

Proof. (i) Since $\mu\text{-}\mathcal{G}s_p\text{int}(A)$ is the greatest $\mu\text{-}\mathcal{G}s_p$-open set containing A and Corollary 2.2 shows that $c^{\Phi_\mu}(i_{p_\mu}(A)) \in \mu\text{-}\mathcal{G}s_pO(X)$. Therefore $c^{\Phi_\mu}(i_{p_\mu}(A)) \subseteq \mu\text{-}\mathcal{G}s_p\text{int}(A)$.

(ii) Since $\mu\text{-}\mathcal{G}s_p\text{cl}(A)$ is the least $\mu\text{-}\mathcal{G}s_p$-closed set containing A and Theorem 2.7(ii) shows that $i_{p_\mu}(c^{\Phi_\mu}(A)) \in \mu\text{-}\mathcal{G}s_pC(X)$. Therefore $\mu\text{-}\mathcal{G}s_p\text{cl}(A) \subseteq i_{p_\mu}(c^{\Phi_\mu}(A))$.

Definition 2.3. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space and (Y, ν) a generalized topological space. A function $f : (X, \mu, \mathcal{G}) \to (Y, \nu)$ is said to be ($\mu\text{-}\mathcal{G}s_p$, ν-pre)-continuous if $f^{-1}(V) \in \mu\text{-}\mathcal{G}s_pO(X)$ for each $V \in \nu PO(Y)$.

Example 2.2. Let $X = \{a, b, c, d\}$, $Y = \{1, 2, 3, 4\}$, $\mu = \{\emptyset, X, \{b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$, $\nu = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ and $\mathcal{G} = P(X) - \{\emptyset, \{c\}\}$. Then $\mu\text{-}\mathcal{G}s_pO(X) = \{\emptyset, X, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $V \in \nu PO(Y) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Define $f : (X, \mu, \mathcal{G}) \to (Y, \nu)$ by $f(a) = 1, f(b) = 2, f(c) = 4$, and $f(d) = 3$. Then inverse image of every ν-preopen sets in Y is $\mu\text{-}\mathcal{G}s_p$-open in X. Hence f is ($\mu\text{-}\mathcal{G}s_p$, ν-pre)-continuous.

Remark 2.3. The concepts of (μ-$\mathcal{G}$-semi, ν)-continuous and ($\mu\text{-}\mathcal{G}s_p$, ν-pre)-continuous are independent.

(i) From Example 2.2, we have that $\mu\text{-}\mathcal{G}SO(X) = \{\emptyset, X, \{b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ and the function f is ($\mu\text{-}\mathcal{G}s_p$, ν-pre)-continuous. Also $f^{-1}(\{1, 2\}) = \{a, b\}$ is not μ-$\mathcal{G}$-semiopen in X for the ν-open set $\{1, 2\}$ of Y. Hence f is not (μ-$\mathcal{G}$-semi, ν)-continuous.

(ii) Let $X = \{a, b, c, d\}$, $Y = \{1, 2, 3, 4\}$, $\mu = \{\emptyset, X, \{b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$, $\nu = \{\emptyset, Y, \{1, 2, 3\}, \{1, 2, 4\}\}$ and $\mathcal{G} = P(X) - \{\emptyset, \{c\}\}$. Then $\mu\text{-}\mathcal{G}s_pO(X) = \{\emptyset, X, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $\nu PO(Y) = \{\emptyset, Y, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$. Define $f : (X, \mu, \mathcal{G}) \to (Y, \nu)$ by $f(a) = 3, f(b) = 2, f(c) = 4$, and $f(d) = 1$. Then the function f is (μ-$\mathcal{G}$-semi, ν)-continuous. Also the inverse image $f^{-1}(\{1\}) = \{d\}$ is not $\mu\text{-}\mathcal{G}s_p$-open in X for the ν-preopen set $\{1\}$ of Y. Hence f is not ($\mu\text{-}\mathcal{G}s_p$, ν-pre)-continuous.

From (i) and (ii), we got the concepts of (μ-$\mathcal{G}$-semi, ν)-continuous and ($\mu\text{-}\mathcal{G}s_p$, ν-pre)-continuous are independent.

Theorem 2.12. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space and (Y, ν) a generalized topological space. For a function $f : (X, \mu, \mathcal{G}) \to (Y, \sigma)$, the following statements are equivalent:
(i) f is (μ-$\mathcal{G}s_p$, ν-pre)-continuous,
(ii) for each $F \in \nu PC(Y)$, $f^{-1}(F) \in \mu$-$\mathcal{G}s_pC(X)$,
(iii) for each $x \in X$ and each $V \in \nu PO(Y)$ containing $f(x)$, there exists a set $U \in \mu$-$\mathcal{G}s_pO(X)$ containing x such that $f(U) \subseteq V$.

Proof. (i) $\Leftrightarrow$ (ii). It is obvious.

(i) $\Rightarrow$ (iii). Let $V \in \nu PO(Y)$ and $f(x) \in V (x \in X)$. Then by (i), $f^{-1}(V) \in \mu$-$\mathcal{G}s_pO(X)$ containing x. Taking $f^{-1}(V) = U$, we have that $x \in U$ and $f(U) \subseteq V$.

(iii) $\Rightarrow$ (i). Let $V \in \nu PO(Y)$ and $x \in f^{-1}(V)$. Then $f(x) \in V \in \nu PO(Y)$ and hence by (iii), there exists a set $U \in \mu$-$\mathcal{G}s_pO(X)$ containing x such that $f(U) \subseteq V$. Now $x \in U \subseteq c^{\Phi_\mu}(i_{p_\mu}(U)) \subseteq c^{\Phi_\mu}(i_{p_\mu}(f^{-1}(V)))$. This shows that $f^{-1}(V) \subseteq c^{\Phi_\mu}(i_{p_\mu}(f^{-1}(V)))$. Thus f is (μ-$\mathcal{G}s_p$, ν-pre)-continuous.

Theorem 2.13. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space and (Y, ν) a generalized topological space. A function $f : (X, \mu, \mathcal{G}) \to (Y, \nu)$ is (μ-$\mathcal{G}s_p$, ν-pre)-continuous if and only if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is (μ-$\mathcal{G}s_p$, ν-pre)-continuous.

Proof. Suppose that f is (μ-$\mathcal{G}s_p$, ν-pre)-continuous. Let $x \in X$ and $W \in (\mu, \nu)$-$PO(X \times Y)$ containing $g(x)$. Then there exists a $U \in \mu PO(X)$ and $V \in \nu PO(Y)$ such that $g(x) = (x, f(x)) \in U \times V \subseteq W$. Since f is (μ-$\mathcal{G}s_p$, ν-pre)-continuous, there exists a $G \in \mu$-$\mathcal{G}s_pO(X)$ containing x such that $f(G) \subseteq V$. By Theorem 2.4(ii), $G \cap U \in \mu$-$\mathcal{G}s_pO(X)$ and $g(G \cap U) \subseteq U \times V \subseteq W$. This shows that g is (μ-$\mathcal{G}s_p$, ν-pre)-continuous. Conversely, suppose that g is (μ-$\mathcal{G}s_p$, ν-pre)-continuous. Let $x \in X$ and $V \in \nu PO(Y)$ containing $f(x)$. Then $X \times V \in (\mu, \nu)$-$PO(X \times Y)$ and by (μ-$\mathcal{G}s_p$, ν-pre)-continuity of g, there exists a $U \in \mu$-$\mathcal{G}s_pO(X)$ containing x such that $g(U) \subseteq X \times V$. Thus we have that $f(U) \subseteq V$ and hence f is (μ-$\mathcal{G}s_p$, ν-pre)-continuous.

Definition 2.4. Let (X, μ) be a generalized topological space and $(Y, \nu, \mathcal{G})$ a grill generalized topological space. A function $f : (X, \mu) \to (Y, \nu, \mathcal{G})$ is said to be (μ-pre, ν-$\mathcal{G}s_p$)-open (resp. (μ-pre, ν-$\mathcal{G}s_p$)-closed) if for each $U \in \mu PO(X)$ (resp. for each $U \in \mu PC(X)$, $f(U)$ is ν-$\mathcal{G}s_p$-open (resp. ν-$\mathcal{G}s_p$-closed) in $(Y, \nu, \mathcal{G})$.

Theorem 2.14. Let (X, μ) be a generalized topological space and $(Y, \nu, \mathcal{G})$ a grill generalized topological space. A function $f : (X, \mu) \to (Y, \nu, \mathcal{G})$ is (μ-pre, ν-$\mathcal{G}s_p$)-open if and only if for each $x \in X$ and each μ-preneighborhood U of x, there exists a $V \in \nu$-$\mathcal{G}s_pO(Y)$ such that $f(x) \in V \subseteq f(U)$.

Proof. Suppose that f is a (μ-pre, ν-$\mathcal{G}s_p$)-open function and let $x \in X$. Also let U be any μ-preneighborhood of x. Then there exists $G \in \mu PO(X)$ such that $x \in G \subseteq U$. Since f is (μ-pre, ν-$\mathcal{G}s_p$)-open, $f(G) = V$ (say) $\in \nu$-$\mathcal{G}s_pO(Y)$ and $f(x) \in V \subseteq f(U)$. Conversely, suppose that $U \in \mu PO(X)$. Then for each $x \in U$, there exists a set $V_x \in \nu$-$\mathcal{G}s_pO(Y)$ such that $f(x) \in V_x \subseteq f(U)$. Thus $f(U) = \bigcup\{V_x : x \in U\}$ and hence by Theorem 2.5(i), $f(U) \in \nu$-$\mathcal{G}s_pO(Y)$. This shows that f is (μ-pre, ν-$\mathcal{G}s_p$)-open.

Theorem 2.15. Let (X, μ) be a generalized topological space, $(Y, \nu, \mathcal{G})$ a grill generalized topological space and let $f : (X, \mu) \to (Y, \nu, \mathcal{G})$ be a (μ-pre, ν-$\mathcal{G}s_p$)-open function. If $V \subseteq Y$ and $F \in \mu PC(X)$ containing $f^{-1}(V)$, then there exists a set $H \in \mu$-$\mathcal{G}s_pO(Y)$ containing V such that $f^{-1}(H) \subseteq F$.

Proof. Suppose that f is (μ-pre, ν-$\mathcal{G}s_p$)-open. Let $V \subseteq Y$ and $F \in \mu PC(X)$ containing $f^{-1}(V)$. Then $X - F \in \mu PO(X)$ and by (μ-pre, ν-$\mathcal{G}s_p$)-openness of f, $f(X - F) \in \mu$-$\mathcal{G}s_pO(Y)$. Thus $H = Y - f(X - F) \in \mu$-$\mathcal{G}s_pC(Y)$ consequently $f^{-1}(V) \subseteq F$ implies that $V \subseteq H$. Further, we obtain that $f^{-1}(H) \subseteq F$.

Theorem 2.16 Let (X, μ) be a generalized topological space and $(Y, \nu, \mathcal{G})$ a grill generalized topological space. For any bijection $f : (X, \mu) \to (Y, \nu, \mathcal{G})$ the following statements are equivalent:
(i) $f^{-1} : (Y, \nu, \mathcal{G}) \to (X, \mu)$ is (ν-$\mathcal{G}s_p$, μ-pre)-continuous,
(ii) f is (μ-pre, ν-$\mathcal{G}s_p$)-open,
(iii) f is (μ-pre, ν-$\mathcal{G}s_p$)-closed.

Proof. It is obvious.

Definition 2.5. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space and a subset A of X is said to be a μ-$\mathcal{G}S^*$-set if $A = U \cap V$, where $U \in \mu PO(X)$, $V \subseteq X$ and $c^{\Phi_\mu}(i_{p_\mu}(V)) = i_{p_\mu}(V)$.

Theorem 2.17. Let $(X, \mu, \mathcal{G})$ be a grill topological space and let $A \subseteq X$. Then $A \in \mu PO(X)$ if and only if $A \in \mu$-$\mathcal{G}s_pO(X)$ and A is a μ-$\mathcal{G}S^*$-set in $(X, \mu, \mathcal{G})$.

Proof. Let $A \in \mu PO(X)$. Then $A \in \mu$-$\mathcal{G}s_pO(X)$, implies that $A \subseteq c^{\Phi_\mu}(i_{p_\mu}(A))$. Also A can be expressed as $A = A \cap X$, where $A \in \mu PO(X)$ and $c^{\Phi_\mu}(i_{p_\mu}(X)) = i_{p_\mu}(X)$. Thus A is a μ-$\mathcal{G}S^*$-set. Conversely, let $A \in \mu$-$\mathcal{G}s_pO(X)$ and A be a μ-$\mathcal{G}S^*$-set. Thus $A \subseteq c^{\Phi_\mu}(i_{p_\mu}(A)) = c^{\Phi_\mu}(i_{p_\mu}(U \cap V))$, where $U \in \mu PO(X)$ and $c^{\Phi_\mu}(i_{p_\mu}(V)) = i_{p_\mu}(V)$. Now $A \subseteq U \cap A \subseteq U \cap c^{\Phi_\mu}(i_{p_\mu}(U \cap V)) = U \cap c^{\Phi_\mu}(U \cap i_{p_\mu}(V)) \subseteq U \cap c^{\Phi_\mu}(U) \cap c^{\Phi_\mu}(i_{p_\mu}(V)) = U \cap i_{p_\mu}(V) = i_{p_\mu}(A)$. Hence $A \in \mu PO(X)$.

Definition 2.6. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space and (Y, ν) a generalized topological space. A function $f : (X, \mu, \mathcal{G}) \to (Y, \nu)$ is (μ-$\mathcal{G}S^*$, ν-pre)-continuous if for each $V \in \nu PO(Y)$, $f^{-1}(V)$ is a μ-$\mathcal{G}S^*$-set in $(X, \mu, \mathcal{G})$.

Theorem 2.18. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space and (Y, ν) a generalized topological space. Then for a function $f : (X, \mu, \mathcal{G}) \to (Y, \nu)$, the following statements are equivalent:
(i) f is (μ, ν)-precontinuous,
(ii) f is (μ-$\mathcal{G}s_p$, ν-pre)-continuous and (μ-$\mathcal{G}S^*$, ν-pre)-continuous.

Proof. It is obvious.

Definition 2.7. Let $(X, \mu, \mathcal{G})$ and $(Y, \nu, \mathcal{G}')$ be two grill generalized topological spaces. A function $f : (X, \mu, \mathcal{G}) \to (Y, \nu, \mathcal{G}')$ is said to be (μ-$\mathcal{G}$, ν-$\mathcal{G}'$)s_p-continuous if $f^{-1}(V) \in \mu$-$\mathcal{G}s_pO(X)$ whenever $V \in \nu$-$\mathcal{G}'s_pO(Y)$.

Note that, in the Example 2.2, consider $\mathcal{G}' = P(Y) - \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Then ν-$\mathcal{G}'s_pO(Y)$ $= \nu PO(Y)$. Hence the function f is (μ-$\mathcal{G}$, ν-$\mathcal{G}'$)s_p-continuous.

Remark 2.4. Every (μ-$\mathcal{G}$, ν-$\mathcal{G}'$)s_p-continuous function is (μ-$\mathcal{G}s_p$, ν-pre)-continuous, but the converse need not be true.

Let $X = \{a, b, c, d\}$, $Y = \{1, 2, 3, 4\}$, $\mu = \{\emptyset, X, \{b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$, $\nu = \{\emptyset, Y, \{2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}\}$, $\mathcal{G} = P(X) - \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{G}' = P(Y) - \{\emptyset, \{3\}\}$. Then μ-$\mathcal{G}s_pO(X) = \{\emptyset, X, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$, $\nu PO(Y) = \{\emptyset, Y, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}\}$ and ν-$\mathcal{G}'s_pO(Y) = \{\emptyset, Y, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$. Define $f : (X, \mu, \mathcal{G}) \to (Y, \nu, \mathcal{G}')$ by $f(a) = 3, f(b) = 2, f(c) = 1$, and $f(d) = 4$. Then the function f is (μ-$\mathcal{G}s_p$, ν-pre)-continuous. Also the inverse image $f^{-1}(\{2, 4\}) = \{b, d\}$ is not μ-$\mathcal{G}s_p$-open in X for the ν-$\mathcal{G}'s_p$-open set $\{2, 4\}$ of Y. Hence f is not (μ-$\mathcal{G}$, ν-$\mathcal{G}'$)s_p-continuous.

Definition 2.8. (i) Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space and a subset A of X is said to be a μ-$\mathcal{G}s_p$-neighborhood of a point $x \in X$ if there exists a set $U \in \mu$-$\mathcal{G}s_pO(X)$ such that $x \in U \subseteq A$.

Note that μ-$\mathcal{G}s_p$-neighborhood of x may be replaced by μ-$\mathcal{G}s_p$-open neighborhood of x.

(ii) Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space. $A \subseteq X$ and $p \in X$. Then p is called a μ-$\mathcal{G}s_p$-limit point of A if $U \cap (A - \{p\}) \neq \emptyset$, for any set $U \in \mu$-$\mathcal{G}s_pO(X)$ containing p. The set of all μ-$\mathcal{G}s_p$-limit points of A is called a μ-$\mathcal{G}s_p$-derived set of A and is denoted by μ-$\mathcal{G}s_p\mathrm{d}(A)$. Clearly if $A \subseteq B$ then μ-$\mathcal{G}s_p\mathrm{d}(A) \subseteq \mu$-$\mathcal{G}s_p\mathrm{d}(B)$.

Remark 2.5. From the Definition 2.8(ii), it follows that p is a μ-$\mathcal{G}s_p$-limit point of A if and only if $p \in \mu$-$\mathcal{G}s_p\mathrm{cl}(A - \{p\})$.

Theorem 2.19. Let $(X, \mu, \mathcal{G})$ be a grill generalized topological space. For any $A, B \subseteq X$, the μ-$\mathcal{G}s_p$-derived sets have the following properties:
(i) μ-$\mathcal{G}s_p\mathrm{cl}(A) \supseteq A \cup \mu$-$\mathcal{G}s_p\mathrm{d}(A)$,
(ii) $\cup_i \mu$-$\mathcal{G}s_p\mathrm{d}(A_i) = \mu$-$\mathcal{G}s_p\mathrm{d}(\cup_i A_i)$,
(iii) μ-$\mathcal{G}s_p\mathrm{d}(\mu$-$\mathcal{G}s_p\mathrm{d}(A)) \subseteq \mu$-$\mathcal{G}s_p\mathrm{d}(A)$,
(iv) μ-$\mathcal{G}s_p\mathrm{cl}(\mu$-$\mathcal{G}s_p\mathrm{d}(A)) = \mu$-$\mathcal{G}s_p\mathrm{d}(A))$.

Proof. Follows from the Definition 2.8(ii) and Remark 2.5.

Theorem 2.20. Let $(X, \mu, \mathcal{G})$ and $(Y, \nu, \mathcal{G}')$ be two grill generalized topological spaces. If $f : (X, \mu, \mathcal{G}) \to (Y, \nu, \mathcal{G}')$ is a function, then the following statements are equivalent:
(i) f is $(\mu$-$\mathcal{G}$, ν-$\mathcal{G}')s_p$-continuous,
(ii) for each $x \in X$, the inverse of every ν-$\mathcal{G}'s_p$-neighborhood of $f(x)$ is a μ-$\mathcal{G}s_p$-neighborhood of x,
(iii) for each $x \in X$ and each ν-$\mathcal{G}'s_p$-neighborhood B of $f(x)$, there is a μ-$\mathcal{G}s_p$-neighborhood A of x such that $f(A) \subseteq B$,
(iv) for each $x \in X$ and each set $B \in \nu$-$\mathcal{G}'s_pO(Y)$ contains $f(x)$, there exists a set $A \in \mu$-$\mathcal{G}s_pO(X)$ containing x such that $f(A) \subseteq B$,
(v) $f(\mu$-$\mathcal{G}s_p\mathrm{cl}(A)) \subseteq \nu$-$\mathcal{G}'s_p\mathrm{cl}(f(A))$ holds for every subset A of X,
(vi) for any set $H \in \nu$-$\mathcal{G}'s_pC(Y)$, $f^{-1}(H) \in \mu$-$\mathcal{G}s_pC(X)$.

Proof. (i) $\Rightarrow$ (ii). Let $x \in X$ and B be a ν-$\mathcal{G}'s_p$-neighborhood of $f(x)$. By Definition 2.8(i), there exists $V \in \nu$-$\mathcal{G}'s_pO(Y)$ such that $f(x) \in V \subseteq B$. This implies that $x \in f^{-1}(V) \subseteq f^{-1}(B)$. Since f is $(\mu$-$\mathcal{G}$, ν-$\mathcal{G}')s_p$-continuous, so $f^{-1}(V) \in \mu$-$\mathcal{G}s_pO(X)$. Hence $f^{-1}(B)$ is a μ-$\mathcal{G}s_p$-neighborhood of x.

(ii) $\Rightarrow$ (i). Let $B \in \nu$-$\mathcal{G}'s_pO(Y)$. Put $A = f^{-1}(B)$. Let $x \in A$. Then $f(x) \in B$. Clearly, B (being ν-$\mathcal{G}'s_p$-open) is a ν-$\mathcal{G}'s_p$-neighborhood of $f(x)$. By (ii), $A = f^{-1}(B)$ is a μ-$\mathcal{G}s_p$-neighborhood of x. Hence by Definition 2.8(i), there exists $A_x \in \mu$-$\mathcal{G}s_pO(X)$ such that $x \in A_x \subseteq A$. This implies that $A = \cup_{x \in A} A_x$. By Theorem 2.5(i), we have that $A \in \mu$-$\mathcal{G}s_pO(X)$. Therefore f is $(\mu$-$\mathcal{G}$, ν-$\mathcal{G}')s_p$-continuous.

(i) $\Rightarrow$ (iii). Let $x \in X$ and B be a ν-$\mathcal{G}'s_p$-neighborhood of $f(x)$. Then, there exists $O_{f(x)} \in \nu$-$\mathcal{G}'s_pO(Y)$ such that $f(x) \in O_{f(x)} \subseteq B$. It follows that $x \in f^{-1}(O_{f(x)}) \subseteq f^{-1}(B)$. By (i), $f^{-1}(O_{f(x)}) \in \mu$-$\mathcal{G}s_pO(X)$. Let $A = f^{-1}(B)$. Then it follows that A is μ-$\mathcal{G}s_p$-neighborhood of x and $f(A) = f(f^{-1}(B)) \subseteq B$.

(iii) $\Rightarrow$ (i). Let $U \in \nu$-$\mathcal{G}'s_pO(Y)$. Take $W = f^{-1}(U)$. Let $x \in W$. Then $f(x) \in U$. Thus U is a ν-$\mathcal{G}'s_p$-neighborhood of $f(x)$. By (iii), there exists a μ-$\mathcal{G}s_p$-neighborhood V_x of x such that $f(V_x) \subseteq U$. Thus it follows that $x \in V_x \subseteq f^{-1}(f(V_x)) \subseteq f^{-1}(U) = W$. Since V_x is a μ-$\mathcal{G}s_p$-neighborhood of x, which implies that there exists a $W_x \in \mu$-$\mathcal{G}s_pO(X)$ such that $x \in W_x \subseteq W$. This implies that $W = \cup_{x \in W} W_x$. By Theorem 2.5(i), $W \in \mu$-$\mathcal{G}s_pO(X)$. Thus f is $(\mu$-$\mathcal{G}$, ν-$\mathcal{G}')s_p$-continuous.

(iii) $\Rightarrow$ (iv). We may replace the μ-$\mathcal{G}s_p$-neighborhood of x as μ-$\mathcal{G}s_p$-open neighborhood of x in condition (iii). Straightforward.

(iv) $\Rightarrow$ (v). Let $y \in f(\mu$-$\mathcal{G}s_p\mathrm{cl}(A))$ and any set $V \in \nu$-$\mathcal{G}'s_pO(Y)$ containing y. Then, there exists a point $x \in X$ and a set $U \in \mu\text{-}\mathcal{G}s_pO(X)$ such that $x \in U$ with $f(x) = y$ and $f(U) \subseteq V$. Since $x \in \mu\text{-}\mathcal{G}s_p\mathrm{cl}(A)$, we have that $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A)$. This implies that $y \in \nu\text{-}\mathcal{G}'s_p\mathrm{cl}(f(A))$. Therefore, we have that $f(\mu\text{-}\mathcal{G}s_p\mathrm{cl}(A)) \subseteq \nu\text{-}\mathcal{G}'s_p\mathrm{cl}(f(A))$.

(v) ⇒ (vi). Let $H \in \nu\text{-}\mathcal{G}'s_pC(Y)$. Then $\nu\text{-}\mathcal{G}'s_p\mathrm{cl}(H) = H$. By condition (v), $f(\mu\text{-}\mathcal{G}s_p\mathrm{cl}(f^{-1}(H))) \subseteq \nu\text{-}\mathcal{G}'s_p\mathrm{cl}(f(f^{-1}(H))) \subseteq \nu\text{-}\mathcal{G}'s_p\mathrm{cl}(H) = H$ holds. Therefore $\mu\text{-}\mathcal{G}s_p\mathrm{cl}(f^{-1}(H)) \subseteq f^{-1}(H)$ and thus $f^{-1}(H) = \mu\text{-}\mathcal{G}s_p\mathrm{cl}(f^{-1}(H))$. Hence $f^{-1}(H) \in \mu\text{-}\mathcal{G}s_pC(X)$.

(vi) ⇒ (i). Let $B \in \mu\text{-}\mathcal{G}s_pO(X)$. We take $H = Y - B$. Then $H \in \nu\text{-}\mathcal{G}'s_pC(Y)$. By (iv), $f^{-1}(H) \in \mu\text{-}\mathcal{G}s_pC(X)$. Hence $f^{-1}(B) = X - f^{-1}(Y - B) = X - f^{-1}(H) \in \mu\text{-}\mathcal{G}s_pO(X)$.

Theorem 2.21. Let $(X, \mu, \mathcal{G})$ and $(Y, \nu, \mathcal{G}')$ be two grill generalized topological spaces. If $f : (X, \mu, \mathcal{G}) \to (Y, \nu, \mathcal{G}')$ is a function, then f is $(\mu\text{-}\mathcal{G},\ \nu\text{-}\mathcal{G}')s_p$-continuous if and only if $f(\mu\text{-}\mathcal{G}s_p\mathrm{d}(A)) \subseteq \nu\text{-}\mathcal{G}'s_p\mathrm{cl}(f(A))$, for all $A \subseteq X$.

Proof. Let $f : (X, \mu, \mathcal{G}) \to (Y, \nu, \mathcal{G}')$ be $(\mu\text{-}\mathcal{G},\ \nu\text{-}\mathcal{G}')s_p$-continuous, $A \subseteq X$ and $x \in \mu\text{-}\mathcal{G}s_p\mathrm{d}(A)$. Assume that $f(x) \notin f(A)$ and let V denote a $\nu\text{-}\mathcal{G}'s_p$-neighborhood of $f(x)$. Since f is $(\mu\text{-}\mathcal{G},\ \nu\text{-}\mathcal{G}')s_p$-continuous and by Theorem 2.20(iii), there exists a $\mu\text{-}\mathcal{G}s_p$-neighborhood U of x such that $f(U) \subseteq V$. From $x \in \mu\text{-}\mathcal{G}s_p\mathrm{d}(A)$, it follows that $U \cap A \neq \emptyset$, there exists at least one element $a \in U \cap A$ such that $f(a) \in f(A)$ and $f(a) \in V$. Since $f(x) \notin f(A)$, we have that $f(a) \neq f(x)$. Thus every $\nu\text{-}\mathcal{G}'s_p$-neighborhood of $f(x)$ contains an element $f(a)$ of $f(A)$ different from $f(x)$. Consequently, $f(x) \in \mathcal{G}'s_p\mathrm{d}(f(A))$. Conversely, suppose that f is not $(\mu\text{-}\mathcal{G},\ \nu\text{-}\mathcal{G}')s_p$-continuous. Then by Theorem 2.20(iii), there exists $x \in X$ and a $\nu\text{-}\mathcal{G}'s_p$-neighborhood V of $f(x)$ such that every $\mu\text{-}\mathcal{G}s_p$-neighborhood U of x contains at least one element $a \in U$ for which $f(a) \notin V$. Put $A = \{a \in X : f(a) \notin V\}$. Since $f(x) \in V$, therefore $x \notin A$ and hence $f(x) \notin f(A)$. Since $f(A) \cap (V - \{f(x)\}) = \emptyset$, therefore $f(x) \notin \mathcal{G}'s_p\mathrm{d}(f(A))$. It follows that $f(x) \in f(\mu\text{-}\mathcal{G}s_p\mathrm{d}(A)) - (f(A) \cup \mathcal{G}'s_p\mathrm{d}(f(A))) \neq \emptyset$, which is a contradiction to the given condition.

Theorem 2.22. Let $(X, \mu, \mathcal{G})$ and $(Y, \nu, \mathcal{G}')$ be two grill generalized topological spaces. If $f : (X, \mu, \mathcal{G}) \to (Y, \nu, \mathcal{G}')$ is an injective function, then f is $(\mu\text{-}\mathcal{G},\ \nu\text{-}\mathcal{G}')s_p$-continuous if and only if $f(\mu\text{-}\mathcal{G}s_p\mathrm{d}(A)) \subseteq \mathcal{G}'s_p\mathrm{d}(f(A))$, for all $A \subseteq X$.

Proof. Let $A \subseteq X$, $x \in \mu\text{-}\mathcal{G}s_p\mathrm{d}(A)$ and V be a $\nu\text{-}\mathcal{G}'s_p$-neighborhood of $f(x)$. Since f is $(\mu\text{-}\mathcal{G},\ \nu\text{-}\mathcal{G}')s_p$-continuous, so by Theorem 2.20(iii), there exists a $\mu\text{-}\mathcal{G}s_p$-neighborhood U of x such that $f(U) \subseteq V$. But $x \in \mu\text{-}\mathcal{G}s_p\mathrm{d}(A)$ gives there exists an element $a \in U \cap A$ such that $a \neq x$. Clearly $f(a) \in f(A)$ and since f is injective, $f(a) \neq f(x)$. Thus every $\nu\text{-}\mathcal{G}'s_p$-neighborhood V of $f(x)$ contains an element $f(a)$ of $f(A)$ different from $f(x)$. Consequently, $f(x) \in \mathcal{G}'s_p\mathrm{d}(f(A))$. Therefore, we have that $f(\mu\text{-}\mathcal{G}s_p\mathrm{d}(A)) \subseteq \mathcal{G}'s_p\mathrm{d}(f(A))$. Converse follows from the Theorem 2.21.

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