Existence Result of Advanced Functional Differential Equation

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Abstract. Here, we discuss the application of fixed point theorem to functional differential equations and proved the existence as well as global attractivity of solution.

Keywords. Functional differential equation, Fixed point theorem; Attractive solution.
Mathematics Subjects Classification: 47H10, 47H08, 47H09.

1. Description of the problem

Suppose R be the real line and R_+ be the set of nonnegative real numbers. Let $I_0=[-\delta,0]$ be a closed and bounded interval in R for some real number $\delta>0$ and $J=I_0\cup R$. Let C denote Banach space of continuous real-valued functions ϕ on I_0 with the supremum norm $\|\cdot\|_C$ defined by

$$\|\phi\|_C=\sup_{t\in I_0}|\phi(t)|$$

Since, C is a Banach space with this supremum norm. For a fixed $t\in R_+$, let u_t denote the element of C defined by

$$u_t(\theta)=u(t+\theta),\quad \theta\in[-\delta,0].$$

Space C is called the history of the past interval I_0 for the functional differential equations to describing the past history of the problems in question.

Let $CRB(R_+)$ denote the class of functions $a:R_+\to R-\{0\}$ satisfying the following properties:

(i) a is continuous

(ii) $\lim_{t\to\infty}a(t)=\pm\infty$ and

(iii) $a(0)=1.$

Given a function $\phi \in C$,

Suppose the following advanced functional differential equation,

$$\begin{aligned} &[a(t)u(t)]' = \alpha(t,u(t),u_t,u(u(t))) + \beta(t,u(t),u_t,u(u(t))) \ \text{a.e. } t \in R_+ \\ &u(0)=\phi \end{aligned} \tag{1.1}$$

where, $a \in CRB(R_+), \alpha : R_+ \times R \times C \times R \to R, \beta : R_+ \times R \times C \times R \to R.$ We discuss above nonlinear functional differential equations on unbounded intervals of real line for existence as well as for characterizations of the solution via classical fixed point theorems in Banach spaces

2. Auxiliary Results

Suppose U be a non-empty set and suppose $T : U \to U$. An invariant point under T in U is called a fixed point of T, that is, the fixed points are the solutions of the functional equation $Tu = u$.

we state some fixed point theorems useful for proving main results.

Suppose U be an infinite dimensional Banach space with the norm $\|\cdot\|$. A mapping $Q : U \to U$ is called D-Lipschitz if there is a continuous and nondecreasing function $\phi : R_+ \to R_+$ satisfying

$$\|Qu - Qv\| \le \phi(\|u - v\|)$$

for all $u, v \in U$, where, $\phi(0)=0$. If $\phi(\mathrm{r}) = kr,\ k>0,$ then Q is called Lipschitz with the Lipschitz constant *k*.

Theorem 2.1 (Granas and Dugundji [14]). Let S be a non-empty, closed, convex and bounded subset of the Banach space U and let $Q : S \to S$ be a continuous and compact operator. Then the operator equation $Qu = u$ (2.1)

has a solution in S.

Suppose the following fixed point theorem of Burton [3] .

Theorem2.*2* (Dhage [7]). Let S be a closed, convex and bounded subset of the Banach space U and let $A : U \to U$ and $B : S \to U$ be two operators such that

(a) A is nonlinear D-contraction

(b) B is completely continuous, and

(c) $u = Au + Bv \Rightarrow u \in S$ for all $v \in S$.

Then the operator equation $$Au + Bu = u \tag{2.2}$$
has a solution in S.

Theorem 2.3. (Dhage [11]). Suppose S be a non-empty closed, convex and bounded subset of the Banach Algebra U and let $A : U \to U$ and $B : S \to U$ be two operators such that

(a) A is D-Lipschitz with D-function ψ,

(b) B is completely continuous,

(c) $u = AuBv \Rightarrow u \in S$ for all $v \in S$ and

(d) $M\psi(r) < r$, where $M = \|B(S)\| = \sup\{\|Bu\| : s \in S\}$.

Then the operator equation

$$AuBu = u \tag{2.3}$$

has a solution in S.

3. Characterizations of Solutions

Define a standard supremum norm $\|\cdot\|$ and a multiplication "." in $BC(I_0 \cup R_+, R)$ by

$$\|u\| = \sup_{t \in I_0 \cup R_+} |u(\mathrm{t})| \text{ and } (uv(t)) = u(t)v(t),\ t \in R_+.$$

We denote the space of Lebesgue integrable functions on R_+

and the norm $\|\cdot\|_{L^1}$ in $L^1(R_+, R)$ is defined by

$$\|u\|_{L^1} = \int_0^\infty |u(t)|\, ds.$$

let us assume the $E = BC(I_0 \cup R_+, R)$ and let Ω be a non-empty subset of U. Let $Q : E \to E$ be a operator and consider the following operator equation in *E*,

$$Qu(t) = u(t) \tag{3.1}$$

for all $t \in I_0 \cup R_+$. Below we give different characterizations of the solutions for the operator equation (3.1) in the space $BC(I_0 \cup R_+, R)$.

Definition 3.1. We say that solutions of the operator equation (3.1) are locally attractive if there exists a closed ball $\overline{B_r}(u_0)$ in the space $BC(I_0 \cup R_+, R)$ for some $u_0 \in BC(I_0 \cup R_+, R)$ such that for arbitrary solutions $u = u(t)$ and $v = v(t)$ of equation (3.1) belonging to $\overline{B_r}(u_0)$ we have that

$$\lim_{t\to\infty}\big(u(t) - v(t)\big) = 0 \tag{3.2}$$

In the case when the limit (3.2) is uniform with respect to the set $\overline{B_r}(u_0)$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that

$$\left|u(t) - v(t)\right| \le \varepsilon \tag{3.3}$$

for all $u, v \in \overline{B_r}(u_0)$ being solutions of (4.1) and for $t \ge T$, we will say that solutions of equation (3.1) are uniformly locally attractive on $I_0 \cup R_+$.

Definition 3.2. A solution $u = u(t)$ of equation (3.1) is said to be globally attractive if (3.2) holds for each solution $v = v(t)$ of (3.1) in $BC(I_0 \cup R_+, R)$. In other words, we may say that solutions of the equation (3.1) are globally attractive if for each arbitrary solutions $u = u(t)$ and $v = v(t)$ of equation (3.1) in $BC(I_0 \cup R_+, R)$, the condition (3.2) is satisfied. In the case when the condition (3.2) is satisfied uniformly with respect to the space $BC(I_0 \cup R_+, R)$, i.e., for every $\varepsilon > 0$ there exists $T > 0$ such that the inequality (3.2) is satisfied for all $u, v \in BC(I_0 \cup R_+, R)$ being the solutions of (3.1) and for $t \ge T$, we will say that solutions of equation (3.1) are uniformly globally attractive on $I_0 \cup R_+$.

4. Attractivity Result

We discuss the problem(2.1) for attractivity characterization of the solutions on unbounded interval $I_0 \cup R_+$. We need the following definitions .

Definition 4.1. By a solution for the differential equation (1.1) we mean a function $u \in BC(I_0 \cup R_+, R) \cap AC(R_+, R)$ such that

(i) The function $t \mapsto a(t)u(t)$ is absolutely continuous on R_+, and

(ii) u satisfies the equations in (1.1), where $AC(R_+, R)$ is the space of absolutely continuous real-valued on right half real axis R_+.

Definition 4.2. Function $\alpha : R_+ \times R \times C \times R \to R$ is caratheodory if

(i) $t \mapsto \alpha(t,u,v,u(u))$ is measurable for all $u \in R_+$ and $v \in C$, and

(ii) $(u,v) \mapsto \alpha(t,u,v,u(u))$ is continuous for all $t \in R_+$.

(H_1). There exists a continuous function $\gamma : R_+ \to R_+$ such that

$$\left|\beta(t,u,v,u(u))\right| \le \gamma(t) \ a.e.\ t \in R_+ \text{ for all } u \in R_+ \text{ and } v \in C.$$

Moreover, we assume that $\lim_{t\to\infty} \left|\overline{a}(t)\right| \int_0^t \lambda(s)ds = 0.$

(H_2). $\phi(0) \ge 0.$

Remark 4.1. If the hypothesis (H_1) holds and $a \in CRB(R_+)$, then $\overline{a} \in BC(R_+, R)$ and the function $w : R_+ \to R$ defined by the expression $w(t) = \left|\overline{a}(t)\right| \int_0^t \gamma(s)ds$ is continuous on R_+. Thus, the number $W = \sup_{t\ge 0} w(t)$ exists.

Theorem 4.1. Suppose that the hypotheses (H_1) holds. Then the functional differential equation(1.1) has a solution and solutions are uniformly globally attractive on $I_0 \cup R_+$.

Proof. Supp0se Set $U = BC(I_0 \cup R_+, R)$. Define an operator *Q* on *X* by

$$Qu(t) = \begin{cases} \phi(0)\overline{a}(t) + \overline{a}(t)\int_0^t [\alpha(s,u(s),u_s,u(u)) + \beta(s,u(s),u_s,u(u))]ds & \text{if } t \in R_+, \\ \phi(t) & \text{if } t \in I_0. \end{cases} \quad (4.1)$$

We prove that *Q* defines a mapping $Q : U \to U$. Suppose $u \in U$ be arbitrary. Clearly, Qu is a continuous function on $I_0 \cup R_+$. We prove that Qu is bounded on $I_0 \cup R_+$. Therefore, if $t \in R_+$, we obtain:

$$\begin{aligned} |Qu(t)| &\le |\phi(0)||\overline{a}(t)| + |\overline{a}(t)|\int_0^t |[\alpha(s,u(s),u_s,u(u)) + \beta(s,u(s),u_s,u(u))]|ds \\ &\le |\phi(0)||\overline{a}(t)| + |\overline{a}(t)|\int_0^t \gamma(s)ds. \end{aligned}$$

Since $\lim_{t\to\infty}\left|\overline{a}(t)\right|\int_0^t \gamma(s)ds = 0$, the function $w: R_+ \to R$ defined by $w(t)=\left|\overline{a}(t)\right|\int_0^t \gamma(s)ds$ is continuous, there is a constant $W > 0$ such that

$$\sup_{t\geq 0} w(t) = \sup_{t\geq 0}\left|\overline{a}(t)\right|\int_0^t \gamma(s)ds \leq W.$$

Thus,

$$|Qx(t)| \leq |\phi(0)| \; \|\overline{a}\| + W \leq \|\overline{a}\| \, \|\phi\| + W$$

for all $t \in R_+$. Similarly, if $t \in I_0$, then $Qu(t) \leq \|\phi\|$. As a result, we have

$$|Qu| \leq (\|\overline{a}\| + 1)\|\phi\| + W \tag{4.2}$$

for all $u \in U$ and thus, *Q* maps U into U itself. Define a closed ball $\overline{B}_r(0)$ centred at the origin of radius *r*, where $r = (\|\overline{a}\|+1)\|\phi\| + W$. Thus *Q* defines a mapping $Q: U \to \overline{B}_r(0)$. particular $Q:\overline{B}_r(0) \to \overline{B}_r(0)$. We verify that *Q* satisfies all the conditions of Theorem 2.2. Firstly, we show that *Q* is continuous on $\overline{B}_r(0)$. For this,, suppose a fix arbitrarily $\varepsilon > 0$,suppose $\{u_n\}$ be a sequence of points in $\overline{B}_r(0)$ converging to a point $u \in \overline{B}_r(0)$. we get:

$$|(Qu_n)(t)-(Qu)(t)| \;\leq |\overline{a}(t)| \int_0^t \begin{matrix} |\alpha(s,u_n(s),u_n(\theta+s),u_n(u_n)) - \alpha(s,u(s),u(\theta+s),u(u))|\,ds \\ |\beta(s,u_n(s),u_n(\theta+s),u_n(u_n)) - \beta(s,u(s),u(\theta+s),u(u))|\,ds \end{matrix} +$$

$$\leq |\overline{a}(t)| \int_0^t \begin{matrix} [\left|\alpha(s,u_n(s),u_n(\theta+s),u_n(u_n))\right| + |\alpha(s,u(s),u(\theta+s),u(u))| \\ |\left|\beta(s,u_n(s),u_n(\theta+s),u_n(u_n))\right| + |\beta(s,u(s),u(\theta+s),u(u))|]ds \end{matrix}$$

$$\leq 2|\overline{a}(t)|\int_0^t \gamma(s)ds$$

$$\leq 2w(t)$$

Thus, by hypothesis (H1), we observe that there exists a $T > 0$ such that $w(t) \leq \varepsilon \;\; for \; t \geq T$. Hence, for $t \geq T$ from (5.2) derive that

$$|(Qu_n)(t)-(Qu)(t)| \;\leq 2\varepsilon \text{ as } n \to \infty. \tag{4.3}$$

Next, Assume that $t \in [0,T]$. Thus, following arguments similar to Dhage [7],by Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} Qu_n(t) = \lim_{x\to\infty}\left[\phi(0)\overline{a}(t) + \overline{a}(t)\int_0^t (\alpha(s,u_n(s),u_n(\theta+s),u_n(u_n)) + (\beta(s,u_n(s),u_n(\theta+s),u_n(u_n))ds\right]$$
$$= \phi(0)\overline{a}(t) + \overline{a}(t)\int_0^t \lim_{x\to\infty}\left[(\alpha(s,u_n(s),u_n(\theta+s),u_n(u_n)) + (\beta(s,u_n(s),u_n(\theta+s),u_n(u_n))\right]ds$$
$$= Qu(t) \tag{4.4}$$

for all $t\in[0,T]$. Similarly, if $t\in I_0$, then

$$\lim_{n\to\infty} Qu_n(t) = \phi(t) = Qu(t)$$

Hence, $Qu_n \to Qu$ as $n\to\infty$ uniformly on R_+ and hence *Q* is a continuous operator on $\overline{B}_r(0)$ into $\overline{B}_r(0)$.

Then , we prove that *B* is compact operator on $\overline{B}_r(0)$. Or this, it is enough to prove that every sequence $\{Qu_n\}$ in $Q\left(\overline{B}_r(0)\right)$ has a Cauchy subsequence. By hypotheses (B_2) and (B_3),

$$|(Qu_n)(t)| \le |\phi(0)||\overline{a}(t)| + |\overline{a}(t)|\int_0^t |\alpha(s,u_n(s),u_n(\theta+s),u_n(u_n)) + \beta(s,u_n(s),u_n(\theta+s),u_n(u_n))|ds$$
$$\le (\|\overline{a}(t)\|+1)\phi(0) + w(t)$$
$$\le (\|\overline{a}(t)\|+1)\|\phi\| + w(t) \tag{4.5}$$

for all $t\in R_+$. Taking supremum over *t*, we have $|Qu_n| \le (\|\overline{a}\|+1)\|\phi\| + W$

for all $n\in N$. This proves that $\{Qu_n\}$ is a uniformly bounded sequence in $Q\left(\overline{B}_r(0)\right)$.

Then, we prove that $Q\left(\overline{B}_r(0)\right)$ is also an equicontinuous set in U. Suppose $\varepsilon>0$ be given. As $\lim_{t\to\infty} w(t)=0$, there is a real number $T_1>0$ such that $|w(t)|<\frac{\varepsilon}{8}$ for all $t\ge T_1$. Similarly, $\lim_{t\to\infty}\overline{a}(t)=0$, for above $\varepsilon>0$, there is a real number $T_2>0$ such that $|\overline{a}(t)|<\frac{\varepsilon}{8|\phi(0)|}$ for all $t\ge T_2$.Thus, if $T=\max\{T_1,T_2\}$, then $|w(t)|<\frac{\varepsilon}{8}$ and $|\overline{a}(t)|<\frac{\varepsilon}{8|\phi(0)|}$ for all $t\ge T$. Suppose $t,\tau\in I_0\cup R_+$ be arbitrary. If $t,\tau\in I_0$, then by uniform continuity of ϕ on I_0. For above ε we have $\delta_1>0$ which is a function of only ε such that

$$|t-T|<\delta_1 \Rightarrow |Qu_n(t)-Qu_n(\tau)| = |\phi(t)-\phi(\tau)| < \frac{\varepsilon}{4}$$

for all $n \in N$. If $t, \tau \in [0,\mathrm{T}]$, we have

$$|Qu_n(t) - Qu_n(\tau)| \le |\phi(0)|\ |\overline{a}(t) - \overline{a}(\tau)|$$

$$+\left| \begin{array}{l} |\overline{a}(t)| \int_0^t [\alpha(s,u_n(s),u_n(\theta+s),u_n(u_n)) + \beta(s,u_n(s),u_n(\theta+s),u_n(u_n))]ds \\ -\overline{a}(\tau)\int_0^\tau [\alpha(s,u_n(s),u_n(\theta+s),u_n(u_n)) + \beta(s,u_n(s),u_n(\theta+s),u_n(u_n))]ds \end{array} \right|$$

$$\le |\phi(0)|\ |\overline{a}(t) - \overline{a}(\tau)|$$

$$+\left| \begin{array}{l} |\overline{a}(t)| \int_0^t [\alpha(s,u_n(s),u_n(\theta+s),u_n(u_n)) + \beta(s,u_n(s),u_n(\theta+s),u_n(u_n))]ds \\ -\overline{a}(\tau)\int_0^t [\alpha(s,u_n(s),u_n(\theta+s),u_n(u_n)) + \beta(s,u_n(s),u_n(\theta+s),u_n(u_n))])ds \end{array} \right|$$

$$+\left| \begin{array}{l} |\overline{a}(\tau)| \int_0^t [\alpha(s,u_n(s),u_n(\theta+s),u_n(u_n)) + \beta(s,u_n(s),u_n(\theta+s),u_n(u_n))]ds \\ -\overline{a}(\tau)\int_0^\tau [\alpha(s,u_n(s),u_n(\theta+s),u_n(u_n)) + \beta(s,u_n(s),u_n(\theta+s),u_n(u_n))]ds \end{array} \right|$$

$$\le |\phi(0)|\ |\overline{a}(t) - \overline{a}(\tau)| + |\overline{a}(t) - \overline{a}(\tau)|$$

$$+\left| \int_0^t [\alpha(s,u_n(s),u_n(\theta+s),u_n(u_n)) + \beta(s,u_n(s),u_n(\theta+s),u_n(u_n))]ds \right|$$

$$+|\overline{a}(\tau)|\left| \int_0^t [\alpha(s,u_n(s),u_n(\theta+s),u_n(u_n)) + \beta(s,u_n(s),u_n(\theta+s),u_n(u_n))]ds \right|$$

$$+\left| \begin{array}{l} |\overline{a}(\tau)| \int_0^t [\alpha(s,u_n(s),u_n(\theta+s),u_n(u_n)) + \beta(s,u_n(s),u_n(\theta+s),u_n(u_n))]ds \\ -\overline{a}(\tau)\int_0^\tau [\alpha(s,u_n(s),u_n(\theta+s),u_n(u_n)) + \beta(s,u_n(s),u_n(\theta+s),u_n(u_n))]ds \end{array} \right|$$

$$\le |\phi(0)|\ |\overline{a}(t) - \overline{a}(\tau)| + |\overline{a}(t) - \overline{a}(\tau)| \int_0^T \gamma(s)ds + \|\overline{a}\| \left| \int_T^t \gamma(s)ds \right|$$

$$\le |\phi(0)|\ |\overline{a}(t) - \overline{a}(\tau)| + |\overline{a}(t) - \overline{a}(\tau)| \int_0^T \gamma(s)ds + \|\overline{a}\| \left| p(t) - p(\tau) \right|$$

$$\le \left[|\phi(0)| + \|\gamma\|_{L^1} \right]\ |\overline{a}(t) - \overline{a}(\tau)| + |\overline{a}(t) - \overline{a}(\tau)| + \|\overline{a}\| \left| p(t) - p(\tau) \right|$$

where, $p(t) = \int_0^t \gamma(s)ds$ and $\|\gamma\|_{L^1} = \int_0^\infty \gamma(s)ds$.

By uniform continuity of the functions $\overline{a}$ and p on $[0,T]$, for above ε. we have the real numbers δ_2 and δ_3 which are the functions of only ε such that

$$|t-\tau|<\delta_2 \Rightarrow |\overline{a}(t)-\overline{a}(\tau)|<\frac{\varepsilon}{\left[|8\phi(0)|+\|\gamma\|_{L^1}\right]}$$

and

$$|t-\tau|<\delta_3 \Rightarrow |p(t)-p(\tau)|<\frac{\varepsilon}{8\|\overline{a}\|}$$

Suppose $\delta_4=\min\{\delta_2,\delta_3\}$. Then

$$|t-\tau|<\delta_4 \Rightarrow |Qu_n(t)-Qu_n(\tau)|<\frac{\varepsilon}{4}$$

for all $n\in N$. Similarly, if $t\in I_0$ and $\tau\in[0,T]$, then we have

$$|Qu_n(t)-Qu_n(\tau)|\le|Qu_n(t)-Qu_n(0)|+|Qu_n(0)-Qu_n(\tau)|.$$

Consider $\delta_5=\min\{\delta_1,\delta_4\}>0$ which is a function of only ε. Hence by estimated facts, follows that

$$|t-\tau|<\delta_5 \Rightarrow |Qu_n(t)-Qu_n(\tau)|<\frac{\varepsilon}{2}$$

for all $n\in N$.

Again, if $t,\tau>T$, we have a real number $\delta_6>0$ which is a function of only ε such that

$$\begin{aligned}
|Qu_n(t)-Qu_n(\tau)| &\le |\phi(0)|\;|a(t)-a(\tau)| \\
&\quad+\left|\begin{array}{l}\overline{a}(t)\int_0^t[\alpha(s,u_n(s),u_n(\theta+s),u_n(\theta+s))+\beta(s,u_n(s),u_n(\theta+s),u_n(\theta+s))]ds \\ -\overline{a}(t)\int_0^\tau[\alpha(s,u_n(s),u_n(\theta+s),u_n(\theta+s))+\beta(s,u_n(s),u_n(\theta+s),u_n(\theta+s))]ds\end{array}\right| \\
&\le |\phi(0)|\;|a(t)|+|\phi(0)|\;|a(\tau)+w(t)+w(\tau) \\
&<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}
\end{aligned}$$

for all $n\in N$, whenever $|t-\tau|<\delta_6$. Similarly, if $t,\tau\in I_0\cup R_+$ with $t<T<\tau$, ,we have

$$|Qu_n(t)-Qu_n(\tau)|\le|Qu_n(t)-Qu_n(T)|+|Qu_n(T)-Qu_n(\tau)|.$$

Consider $\delta = \min\{\delta_5, \delta_6\} > 0$ which is a function of only ε. Thus by above estimated facts, follows that

$$|Qu_n(t) - Qu_n(T)| < \frac{\varepsilon}{2} \text{ and } |Qu_n(T) - Qu_n(\tau)| < \frac{\varepsilon}{2}$$

for all $n \in N$, $|t-\tau| < \delta$. As , $|Qu_n(t) - Qu_n(\tau)| < \varepsilon$ for all $t, \tau \in I_0 \cup R_+$,for all $n \in N$, $|t-\tau| < \delta$. This proves that $\{Qu_n\}$ is a equicontinuous sequence in U. From Arzela-Ascoli theorem that $\{Qu_n\}$ has a uniformly convergent subsequence on the compact subset $I_0 \cup [0,T]$ of $I_0 \cup R$. Without loss of generality, call subsequence to be the sequence itself. We prove that $\{Qu_n\}$ is Cauchy in U. Now $|Qu_n(t) - Qu(t)| \to 0$ as $n \to \infty$ for all $t \in I_0 \cup [0,T]$. Thus for given $\varepsilon > 0$ there exists an $n_0 \in N$ such that

$$\sup_{-\delta \le p \le T} |\bar{a}(p)| \int_0^p \left| \begin{matrix} [\alpha(s, u_n(s), u_n(\theta+s), u_n(u_n)) + \beta(s, u_n(s), u_n(\theta+s), u_n(u_n))] \\ -[\alpha(s, u(s), u(\theta+s), u(u)) + \beta(s, u(s), u(\theta+s), u(u))] \end{matrix} \right| ds < \frac{\varepsilon}{2}$$

for all $m, n \ge n_0$. Hence, if $m, n \ge n_0$, we have

$$\begin{aligned} \|Qu_m - Qu_n\| &= \sup_{-\delta \le t < \infty} \left| |\bar{a}(t)| \int_0^t \left| \begin{matrix} [\alpha(s, u_m(s), u_m(\theta+s), u_m(u_m)) + \beta(s, u_m(s), u_m(\theta+s), u_m(u_m))] \\ -[\alpha(s, u_n(s), u_n(\theta+s), u_n(u_n)) + \beta(s, u_n(s), u_n(\theta+s), u_n(u_n))] \end{matrix} \right| ds \right| \\ &< \sup_{-\delta \le t < T} \left| |\bar{a}(p)| \int_0^p \left| \begin{matrix} [\alpha(s, u_m(s), u_m(\theta+s), u_m(u_m)) + \beta(s, u_m(s), u_m(\theta+s), u_m(u_m))] \\ -[\alpha(s, u_n(s), u_n(\theta+s), u_n(u_n)) + \beta(s, u_n(s), u_n(\theta+s), u_n(u_n)) \end{matrix} \right| ds \right| \\ &\quad + \sup_{p \ge T} |\bar{a}(p)| \int_0^p \left[\left| \begin{matrix} [\alpha(s, u_n(s), u_n(\theta+s), u_n(u_n)) + \beta(s, u_n(s), u_n(\theta+s), u_n(u_n)) \\ +[\alpha(s, u_n(s), u_n(\theta+s), u_n(u_n)) + \beta(s, u_n(s), u_n(\theta+s), u_n(u_n)) \end{matrix} \right| \right] ds \\ &< \varepsilon. \end{aligned}$$

This proves that $\{Qx_n\} \subset Q(\bar{B}_r(0)) \subset X$ is Cauchy. Since X is complete, $\{Qx_n\}$ converges to a point in X. Since $Q(\bar{B}_r(0))$ is closed $\{Qx_n\}$ converges to a point in $Q(\bar{B}_r(0))$. Hence $Q(\bar{B}_r(0))$is relatively compact .Consequently, Q is a continuous and compact operator on $\bar{B}_r(0)$ into itself. From Theorem 2.2 to the operator Q on

$\overline{B}_r(0)$ gives Q has a fixed point in $\overline{B}_r(0)$ which proves that the problem(1.1) has a solution on $I_0 \cup R_+$.

Finally, to prove that the solutions are uniformly attractive on $I_0 \cup R_+$. Suppose $x, y \in \overline{B}_r(0)$ be any two solutions of the problem (3.1) defined on $I_0 \cup R_+$. Then,

$$|u(t)-v(t)| \le \left| \begin{array}{l} \overline{a}(t)\int_0^t [\alpha(s,u(s),u_s,u(u)) + \beta(s,u(s),u_s,u(u))]ds \\ -\overline{a}(t)\int_0^t [\alpha(s,v(s),v_s,v(v)) + \beta(s,v(s),v_s,v(v))]ds \end{array} \right|$$

$$\le |\overline{a}(t)| \int_0^t \left|[\alpha(s,u(s),u_s,u(u)) + \beta(s,u(s),u_s,u(u))]\right| ds$$

$$+ |\overline{a}(t)| \int_0^t \left|[\alpha(s,v(s),v_s,v(v)) + \beta(s,v(s),v_s,v(v))]\right| ds$$

$$\le 2w(t) \qquad (4.6)$$

for all $t \in I_0 \cup R_+$. Now $\lim_{t\to\infty} w(t) = 0$, then a real number $T > 0$ such that $w(t) < \frac{\varepsilon}{2}$ for all $t \ge T$. Hence, $|u(t)-v(t)| \le \varepsilon$ for all $t \ge T$, so all the solutions of the problem (1.1) are uniformly globally attractive on $I_0 \cup R_+$.

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