Perturbed Functional Random Differential Equation

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Abstract : In this paper the existence of the solution for a first order nonlinear perturbed functional random differential equations is proved by using a Leray Schauder type fixed point theorem.

Keywords: Random differential equation, fixed point, existence of solution.

2000 Mathematics subject Classification:34F05,60H10.

1.Description of the Problems

Let $\mathbb{R}$ denote the real line. Let $I_0 = [-r, 0]$ and $I = [0, a]$ be two closed and bounded intervals in $\mathbb{R}$ for some real numbers r and a with $r > 0$ and $a > 0$. Let $\mathcal{C} = C(I_0, \mathbb{R})$ denote the space of all continuous real valued functions on I_0 equipped with the $\|.\|_c$ defined by

$$\|u\|_{\mathcal{C}} = \sup_{t \in I_0} \|u(t)\|$$

Given a measurable space (Ω, A) and a given a history function $\phi : \Omega \to C(I_0, \mathbb{R})$,

We discuss the following perturbed functional random differential equation (FRDE)

$$\begin{aligned} & u'(t,\omega) = p(t, u_t(\omega), \omega) + q(t, u_t(\omega), \omega) + r(t, u_t(\omega), \omega) \quad a.e.\, t \in I_0 \\ & u(t,\omega) = \phi(t,\omega) \;, t \in I_0 \end{aligned} \qquad \{1.1\}$$

for all $\omega \in \Omega$ where $p, q, r : I \times C \times \Omega$

Here, we shall use a random version of the Leray-Schauder type principle proved in Dhage [3] and study the nonlinear initial value problems of perturbed functional random differential equation of first order of the solution under suitable conditions.

2.Auxiliary results

let (Ω, A) denote a measurable space, U a separable Banach space. Let β_U be a sigma algebra of all Borel subsets of U. A mapping $u:\Omega \to U$ is called measurable if for any $B \in \beta_U$.

$$u^{-1}(B)=\{\omega \in \Omega / u(\omega) \in B\} \in \mathrm{A}$$

We recall that a multi valued mapping $F:\Omega \to 2^u \setminus \phi$ is called measurable if for any in

$B \in \beta_U$.

$$F^{-1}(B)=\{\omega \in \Omega / F(\omega) \cap B \neq \phi\} \in \mathrm{A}$$

A measurable mapping $\psi:\Omega \to U$ is called a measurable selector of the multi-valued napping $F:\Omega \to 2^u \setminus \phi$ if $\psi(\omega) \in F(\omega)$ for $\omega \in \Omega$.

Let $T:U \to U$, T is called a contraction if there exists a constant $\alpha < 1$ such that $\|T_u - T\| \leq \alpha \|u-v\|$ for all $u, v \in U$, T is called compact if $\overline{T(U)}$ is the compact subset of U where $\overline{T(U)}$ is the closure of $T(U)$ inU. T is called totally bounded if for any bounded subset S of U,$T(S)$ is a totally bounded set in U.

A random operator $T:\Omega \times U \to U$ is called contraction (resp. compact. totally bounded and completely continuous) if $T(\omega)$ is contraction (resp. compact, totally bounded and completely continuous) for each $\omega \in \Omega$.

We need the following fixed point theorem of Dhage $[4]$.

Theorem 2.1. Let $A, B, C:\Omega \times U \to U$ be two random operators satisfying for each $\omega \in \Omega$

(a). $A(\omega)$, $B(\omega)$,are contractions.

(b). $C(\omega)$ is completely continuous and

(c). the set $\varepsilon = \{\omega \in \Omega \to U / A(\omega)u + B(\omega)u + C(\omega)u = \alpha u\}$ is bounded for all $\alpha > 1$

Then the random equation

$A(\omega)u + B(\omega)u + C(\omega)u = u$

has a random solution.

Next we prove a random version of the following fixed point theorem of Dhage $[3]$

Theorem2.2. Let $A, B, C : U \to U$ be operators such that:

(a) A ,B are linear and bounded and there exists a $p \in \mathbb{N}$ such that A^p is a nonlinear contraction, and

(b) C is completely continuous.

Then either

(i) the operator equation $Au + Bu + \lambda Cu = u$ has a solution for $\lambda = 1$ or

(ii) the set $\varepsilon = \{u \in U / Au + Bu + \lambda Cu = u, 0 < \lambda < 1\}$ is unbounded.

Theorem2.3. Let $A, B, C : \Omega \times U \to U$ be random operators satisfying for each $\omega \in \Omega$

(a) $A(\omega)$, $B(\omega)$ are linear and bounded, and there exists a $p \in \mathbb{N}$ such that A^p is a nonlinear contraction, and

(b) $C(\omega)$ is completely continuous and

(c) the set $\varepsilon = \{u \in U / A(\omega)u + A(\omega)u + \lambda(\omega)C(\omega)u = u\}$ is bounded for every measurable function $\lambda : \Omega \to \mathbb{R}$ with $0 < \lambda(\omega) < 1$.

Then the operator equation

$$A(\omega)u + B(\omega)u + C(\omega)u = u$$

has a random solution.

As a consequence of Theorem 2.2 we obtain

Corollary 2.4 Let $A, B, C : \Omega \times U \to U$ be two random operators satisfying for each $\omega \in \Omega$

(a) $A(\omega)$, $B(\omega)$ are contractions.

(b) $C(\omega)$ is completely continuous.

(c).the set $\varepsilon = \{u \in U \mid A(\omega)u + B(\omega)u + \lambda(\omega)C(\omega)u = u\}$ is bounded for each $\lambda \in (0,1)$

then the random equation (1.1) has a random solution.

3. Existence Theory

Let $M(J,\mathbb{R}), B(J,\mathbb{R}), BM(J,\mathbb{R}), AC(J,\mathbb{R}), C(J,\mathbb{R})$ denote respectively the spaces of all measurable, bounded, bounded and measurable, absolutely continuous and continuous real valued functions on J .Define a norm $\|.\|_C$ in $C(J,\mathbb{R})$ by

$$\|u\|_C = \max_{t \in J} |u(t)|$$

Clearly $C(J,\mathbb{R})$ is a separable Banach space with this supremum norm. We need the following definitions.

Definition 3.1. A function $\beta : J \times C \times \Omega \to \mathbb{R}$ is said to be ω-Caratheodory if for each $\omega \in \Omega$

(i) $.t \to p(t,u,\omega)$ is measurable for all $u \in C$, and

(ii) $u \to p(t,u,\omega)$ is continuous for almost everywhere $t \in J$

Further a ω-Caratheodory function β is called L^1_ω-Caratheodory if

(iii) for each real number $k > 0$ there exists a function $h_k : \Omega \to L^1_\omega(J,\mathbb{R})$ such that

$|\beta(t,u,\omega)| \le h_k(t,\omega), a.e. t \in J$

For all $u \in C$ with $\|u(\omega)\|_C \le k$

We consider the following set of hypotheses.

(A). The function $\omega \to p(t,u,\omega)$, $\omega \to q(t,u,\omega)$ are measurable for all $t \in I$ and $u \in C$

(B). The function $t \to p(t,u,\omega)$, $q \to p(t,u,\omega)$ are continuous for each $\omega \in \Omega$, and there exists a function $\alpha : \Omega \to L^1(J,\mathbb{R})$, with $\|\alpha(\omega)\|_{L^1} < 1$, such that for each $\omega \in \Omega$

$$|p(t,u,\omega) - p(t,v,\omega)| \le \alpha(t,\omega)\|u(\omega) - v(\omega)\|_C \ a.e. t \in I,$$

$$|q(t,u,\omega) - q(t,v,\omega)| \le \alpha(t,\omega)\|u(\omega) - v(\omega)\|_C \ a.e. t \in I$$

for all $u, v \in C$.

(C) .The function $\omega \to r(t,u,\omega)$ is measurable for all $t \in I$ and $u \in C$

(D).The function g is L^1_ω- Caratheodory.

(E).There exists a function $\gamma : \Omega \to L^1(J,\mathbb{R})$ with $\gamma(t,\omega) > 0$ a.e. $t \in J$ and a continuous nondecreasing function $\psi : (0,\infty] \to (0,\infty)$ such that

$$|r(t,u,\omega)| \le \gamma(t,\omega)\psi\left(\|u(\omega)\|_C\right) a.e. t \in I$$

for all $u \in \mathbb{R}$.

Theorem 3.1. Assume that hypothesis (A) -(E) hold. Further suppose that $\|\alpha(\omega)\|_{L^1} < 1$ and

$$\int_0^\infty \frac{dw}{w + \chi(w)} > \left\|\tilde{\chi}(\omega)\right\|_{L^1} \tag{3.1}$$

Where

$c_0(\omega) = \|\phi(\omega)\|_{\mathbb{R}} + \int_0^t |r(s,0,\omega)| ds$ and $\tilde{\chi}(s,\omega) = \max\{\alpha(s,\omega), \chi(s,\omega)\}$

Then the perturbed FRDE (1.1) has a solution on J.

Proof. Let $U = \square(J,R)$, Now FRDE (1.1) is equivalent to the random integral equation (RIE)

$$u(t,\omega) = \phi(0,\omega) + \int_0^t p(t,u_s(\omega),\omega)ds + \int_0^t q(t,u_s(\omega),\omega)ds + \int_0^t r(t,u_s(\omega),\omega)ds,$$

$$a.e.\ t \in I \qquad (3.2)$$

$$= \phi(t,\omega), \qquad t \in I_0$$

Define three operators $A,B,C : J \times C \times \Omega \to U$ by

$$A(\omega)u(t,\omega) = \int_0^t p(t,u_s(\omega),\omega)ds \qquad a.e.\ t \in I,$$

$$= 0 \qquad t \in I_0$$

and

$$B(\omega)u(t,\omega) = \int_0^t q(t,u_s(\omega),\omega)ds, \qquad a.e.\ t \in I,$$

$$= 0 \qquad t \in I_0$$

$$C(\omega)u(t,\omega) = \int_0^t r(t,u_s(\omega),\omega)ds$$

$$= \phi(t,\omega)$$

Then the problem of finding the random solution of the perturbed FRDE (1.1) is just reduced to finding the random solution of random equation $A(\omega)u(t,\omega) + B(\omega)u(t,\omega) + C(\omega)u(t,\omega) = u(t,\omega)$ $t \in I$ in U. This further implies that the random fixed points of the operator equation $A(\omega)u(t,\omega) + B(\omega)u(t,\omega) + C(\omega)u(t,\omega) = u(t,\omega)$ are the random solution of the FRDE (1.1) on J. We shall show that the operators $A(\omega), B(\omega)$ *and* $C(\omega)$ satisfying all the condition of Theorem 2.1.

Step I: First we show that $A(\omega)$, $B(\omega)$ and $C(\omega)$ are random operators on U. Since

$$\omega \mapsto q\left(t, u_t(\omega), \omega\right)$$

Is measurable for each $t \in I$ and $u \in \square$,and the integral on the right hand side of the equation (3.2) is the limit of the finite sum of measurable function, the function

$$\omega \mapsto \int_0^t r\left(t, u_s(\omega), \omega\right) ds$$

and

Is measurable. Hence the operator $A(\omega)$ is a random operator on U .

Again the function $\omega \to \phi(t, \omega)$ is measurable for each $t \in I_0$ and the integral

$$\omega \mapsto \int_0^t r\left(t, u_s(\omega), \omega\right) ds$$

Is measurable, therefore and the sum $\phi(0, \omega) + \int_0^t r\left(t, u_s(\omega), \omega\right) ds$ is measurable in $\omega \in \Omega$ for each $t \in I$ .Hence the operator $C(\omega)$ is a random operator on U .

Step II: Next we show that $A(\omega)$ is a contraction random operator on U . Let $u, v \in U$. Then by(H2)

$$\left|A(\omega)u(t) - A(\omega)v(t)\right| = \int_0^t p\left(s, u_t(\omega), \omega\right) ds - \int_0^t p\left(t, v_t(\omega), \omega\right) ds$$

$$\leq \alpha(t, \omega) \left\| u_t(\omega) - v_t(\omega) \right\|_{\square}$$

$$\leq \left\| \alpha(\omega) \right\|_{L^1} \left\| u(\omega) - v(\omega) \right\|_{\square}$$

Taking supremum over t, we obtain

$$\left|A(\omega)u(t) - A(\omega)v(t)\right| \leq \left\| \alpha(\omega) \right\|_{L^1} \left\| u(\omega) - v(\omega) \right\|_{\square}$$

for all $u, v \in U$ and $\omega \in \Omega$ where $\left\| \alpha(\omega) \right\|_{L^1} < 1$. This shows that $A(\omega)$ is a contraction random operator on U .

Similarly,we can show that $B(\omega)$ is a contraction random operator on U . Let $u,v\in U$. Then by(B)

$$\left|B(\omega)u(t)-B(\omega)v(t)\right|=\int_0^t q\left(s,u_t(\omega),\omega\right)ds-\int_0^t q\left(t,v_t(\omega),\omega\right)ds$$

$$\leq \beta(t,\omega)\left\|u_t(\omega)-v_t(\omega)\right\|_{\mathbb{C}}$$

$$\leq \left\|\beta(\omega)\right\|_{L^1}\left\|u(\omega)-v(\omega)\right\|_{\mathbb{C}}$$

Taking supremum over t, we obtain

$$\left|B(\omega)u(t)-B(\omega)v(t)\right|\leq \left\|\beta(\omega)\right\|_{L^1}\left\|u(\omega)-v(\omega)\right\|_{\mathbb{C}}$$

for all $u,v\in U$ and $\omega\in\Omega$ where $\left\|\beta(\omega)\right\|_{L^1}<1$. This shows that $B(\omega)$ is a contraction random operator on U .

Step III: Now we shall show that the random operator $C(\omega)$ is completely continuous on U . First we show that $C(\omega)$ is continuous on U **.**Using the dominated convergence theorem and the continuity of the function $r(t,u,\omega)$ in u ,it fallows that

$$C(\omega)u_n(t,\omega)=\phi(\theta,\omega)+\int_0^t r\left(s,u_n(s+\theta,\omega),\omega\right)ds$$

$$=\phi(\theta,\omega)+\int_0^t r\left(s,u(s+\theta,\omega),\omega\right)ds$$

$$=C(\omega)u(t,\omega)$$

for all $t\in I$.

Similarliy,

$$\left|C(\omega)u_n(t,\omega)\right|=\phi(t,\omega)=C(\omega)u(t,\omega) \text{ forall } t\in I_\theta$$

This shows that the $C(\omega)$ is continuous random operator on U .

Next we show that $C(\omega)$ is totally bounded random operator on U .To finish, it is enough to prove that $\{C(\omega)u_n : n\in N\}$ is uniformly bounded and equicontinuous set in U . Suppose that $u_n(t,\omega)$ is a bounded sequence in U . Then there is a real number $s>0$ such that $u_n(t,\omega)\le s, \forall n\in N$.

$$|C(\omega)u_n(t,\omega)| \le \max\{|\phi(\theta,\omega)|,|\phi(t,\omega)|\} + \int_0^t |r(s,u_n(s+\theta,w),\omega)| ds$$

$$\le \|\phi(\omega)\|_{\square} + \int_0^t h_s(s,\omega)$$

$$\le \|\phi(\omega)\|_{\square} + \int_0^a h_s(s,\omega)$$

$$\le \|\phi(\omega)\|_{\square} + \|h_s(\omega)\|_{L^1}$$

Taking supremum over t, we obtain

$$|C(\omega)u_n| \le \|\phi(\omega)\|_{\square} + \|h_s(\omega)\|_{L^1}$$

Which shows that $\{C(\omega)u_n : n\in N\}$ is uniformly bounded set in U .

Next we show that the set $\{C(\omega)u_n : n\in N\}$ is an equicontinuous set. Let t, $\tau\in I$

Then

$$|C(\omega)u(t) - C(\omega)u(\tau)| < \left|\int_u^t r(s,u_n(\omega),\omega)ds - \int_0^\tau r(s,x_n(\omega),\omega)ds\right|$$

$$\le \left|\int_\tau^t |r(s,u_n(\omega),\omega)| ds\right|$$

$$\le \left|\int_\tau^t h_s(s,\omega)ds\right|$$

$$\le |a(t,\omega) - a(\tau,\omega)|$$

Where $a(t,\omega)=\int_u^t h_s(s,\omega)ds$

Since p is continuous on *I,* it is uniformly continuous on *I.* Therefore

$$\left|C(\omega)u(t)-C(\omega)u(\tau)\right|\to 0 \text{ as } t\to\tau$$

Again let $t,\tau\in I_\theta$ Then we have

$$\left|C(\omega)u(t)-C(\omega)u(\tau)\right|=\left|\phi(t,\omega)-\phi(\tau,\omega)\right|\to 0 \text{ as } t\to\tau$$

Similarly if $t\in I$ and $\tau\in I_0$ then we obtain

$$\left|C(\omega)u(t)-C(\omega)u(\tau)\right|=\phi(\tau,\omega)-\phi(0,\omega)-\left|\int_0^t r\left(s,u_n(\omega),\omega\right)ds\right|$$

$$\le\left|\phi(t,\omega)-\phi(0,\omega)\right|+\left|\int_0^t r\left(s,u_n(\omega),\omega\right)ds\right|$$

$$\le\left|\phi(t,\omega)-\phi(0,\omega)\right|+\int_0^t\left|r\left(s,u_n(\omega),\omega\right)\right|ds$$

$$\le\left|\phi(t,\omega)-\phi(0,\omega)\right|+\int_0^t h_s(s,\omega)ds$$

Now if $|t-\tau|\to 0,$ thus we have $t\to 0$ as $\tau\to 0$,so by continuity of ϕ and the integral, it follows that

$$\left|C(\omega)u(t)-C(\omega)u(\tau)\right|\to 0 \text{ as } t\to\tau$$

Hence the set $\{C(\omega)u_n : n\in N\}$ is an equicontinuous in U. Thus the random operator $C(\omega)$ is completely continuous in view of Arezela-Ascoli Theorem.

Finally we show that the hypothesis (c) of Theorem 2.1 holds.

Let $l\in\varepsilon$ be arbitrary. Then we have

$$A(\omega)l(t,\omega)+B(\omega)l(t,\omega)+C(\omega)l(t,\omega)=\lambda l(t,\omega) \;,\lambda>1 \text{ for}$$

all $t \in J$.Therefore

$$l(t,\omega) = \lambda^{-1}\left[A(\omega)l(t) + B(\omega)l(t) + C(\omega)l(t)\right]$$

for $t \in J$.Hence

$$|l(t,\omega)| =$$

$$= \lambda^{-1}\begin{cases} \varphi(0,\omega) + \int_0^t p(t,l_s(\omega),\omega)\,ds + \int_0^t q(t,l_s(\omega),\omega)\,ds + \int_0^t r(t,l_s(\omega),\omega)\,ds, t \in I \\ \varphi(t,\omega), t \in I_0 \end{cases}$$

Hence if $t \in I$,

$$|l(t,\omega)| \le |\lambda^{-1}| \max\{|\phi(0,\omega)|, |\phi(t,\omega)|\} + |\lambda^{-1}|\left|\int_0^t p(s,l_s(\omega),\omega)\,ds\right| + |\lambda^{-1}|\left|\int_0^t q(s,l_s(\omega),\omega)\,ds\right|$$

$$+|\lambda^{-1}|\left|\int_0^t r(s,l_s(\omega),\omega)\,ds\right|$$

$$\le \|\phi(\omega)\|_C + \int_0^t |p(s,l_s(\omega),\omega)|ds + \int_0^t |q(s,l_s(\omega),\omega)|ds + \int_0^t |r(s,l_s(\omega),\omega)|ds$$

$$\le \|\phi(\omega)\|_{\square} + \int_0^t |p(s,l_s(\omega),\omega) - p(s,0,\omega)|ds + \int_0^t |p(s,0,\omega)| + \int_0^t \gamma(t,\omega)\varphi(\|l_s(\omega)\|_{\square})\,ds$$

$$\le \|\phi(\omega)\|_{\square} + \int_0^t \alpha(s,\omega)\|l_s(\omega)\|_{\square})ds + \int_0^t |p(s,0,\omega)|ds + \int_0^t \gamma(t,\omega)\varphi(\|l_s(\omega)\|_{\square})\,ds$$

$$\le c_0(\omega) + \int_0^t \hat{\gamma}(s,\omega)\left[\|l_s(\omega)\|_{\square} + \varphi(\|l_s(\omega)\|_{\square})\right]\text{ds}$$

Set $l(t,\omega) = \max_{s\in[-r,t]}|l(s,\omega)|$ Then $|u(t,\omega)| \le w(t,\omega), \forall t \in J$ and $\omega \in \Omega$, and there is a $t^* \in [-r,t]$ such that

$$w(t,\omega) = |l(t^*,\omega)| = \max_{s\in[-r,t]}|l(s,\omega)|$$

for all $\omega \in \Omega$ .Therefore for any $t \in I$ we get

$$l(t,\omega)=c_0(\omega)+\int_0^t \hat{\gamma}(s,\omega)\|l_s(\omega)\|_{\square}+\varphi\left(\|l_s(\omega)\|_{\square}\right)ds$$

$$\leq c_0(\omega)+\int_0^t \hat{\gamma}(s,\omega)\left[w(s,\omega)+\varphi\left(w(s,\omega)\right)\right]ds.$$

Let

$$m(t,s)=c_0(\omega)+\int_0^t \hat{\gamma}(s,\omega)\left[w(s,\omega)+\varphi\left(w(s,\omega)\right)\right]ds, t\in I$$

Then we have $w(t,\omega)\leq m(t,\omega), \forall t\in I$ and $\omega\in\Omega$ and $m(0,\omega)=c_0(\omega)$. Differentiating w.r.t. t yields

$$m'(t,\omega)=\hat{\gamma}(t,\omega)\left[w(t,\omega)+\varphi\left(w(t,\omega)\right)\right]$$

$$\leq \hat{\gamma}(t,\omega)\left[m(t,\omega)+\varphi\left(m(t,\omega)\right)\right], t\in I$$

Hence from above inequality we obtain

$$\frac{m'(t,\omega)}{m(t,\omega)+\varphi\left(m(t,\omega)\right)}\leq \hat{\gamma}(t,\omega), t\in I .$$

Integrating from 0 to t gives

$$\int_0^t \frac{m'(t,\omega)}{m(t,\omega)+\varphi\left(m(t,\omega)\right)}ds\leq \int_0^t \hat{\gamma}(t,\omega)\,ds$$

By change of the variable, we obtain

$$\int_{c_0(\omega)}^{m(s,\omega)} \frac{dw}{w+\varphi(w)}\leq \int_0^t \hat{\gamma}(s,\omega)\,ds\leq \int_0^a \hat{\gamma}(s,\omega)\,ds<\int_{c_0(\omega)}^{\infty}\frac{dw}{w+\varphi(w)}$$

This implies that there exists a constant $M(\omega)>0$ such that

$$m(t,\omega)<M(\omega), \forall t\in J \text{ and } \omega\in\Omega$$

Then we have

$$\left|u(t,\omega)\right| \le \left|w(t,\omega)\right| \le \left|u(t,\omega)\right| \le M(\omega), \forall t \in J \text{ and } \omega \in \Omega$$

Then the set ε is bounded. Hence an application of Theorem 2.1 yields that the perturbed FRDE (1.1) has a solution on J.

References

1. T. A. Burton and C. Kirk, A fixed point theorem of Krasnoselskri-Schaefer Type, Math, Nachr, **198**(1998), 23-31.
2. B. C. Dhage, Monotone iterative technique for Caratheodory theory of nonlinear functional random integral equations, Thamkang J. Math **33**(2002), 341 351.
3. B. C. Dhage, J. Henderson. It. N. Ingle and S. M. Kang. Nonlinear perturbed random boundary value problems, Comm. Appl. Nonlinear Anal. 11 (2004), 91-107.
4. S. Heikkila and V. Lakshmikantham, Monotone Iterative Technique for Nonlinear Discontin sour Differential Equations, Marcel Dekker Inc., New York, 1994.
5. S. Itoh, Random fired point theorems with applications to random differential equations in Banack spaces, J. Math. Anal Appl 67 (1979), 261-273.
6. M. A. Krasnoselskil, Topological fethods in the Theory of Nonlinear Integral Equations, Cambridge University Press, London, 1964.