Analytical and Numerical Solutions of a Parabolic Equation with Function-Exponent Nonlinearities

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Abstract

This study investigates the analytical and numerical solutions of a parabolic differential equation with function-exponent nonlinearities using the Akbari-Ganji method (AGM) and numerical techniques implemented in Scilab. The equation under consideration models various physical and engineering phenomena, including heat conduction, reaction-diffusion systems, and electrochemical processes. The AGM approach is applied to obtain an approximate closed-form solution by transforming the nonlinear DE into a solvable form while minimizing residual errors. The equation is computed and solved using Scilab's finite difference approach for numerical validation.The analytical and numerical results are compared to assess accuracy, convergence, and computational efficiency. The findings highlight the effectiveness of AGM in handling nonlinearities while demonstrating the reliability of Scilab-based numerical schemes. This study provides insights into solving complex parabolic DEs with nonlinear exponent functions, offering applications in various scientific and engineering fields.

Keywords: Nonlinear DE, Function-Exponent nonlinearity, Analytical methods, Numerical simulation

Mathematics Subject Classification 35B44 · 35D30 · 35L70

1 Introduction

Researchers have become interested in studying partial differential equations with exponent nonlinearities in the last few years. Scientists are interested in these kinds of problems because they are used in fluid mechanics, hydrodynamics, complex elasticity, electrorheological fluids, and other fields [1]. Parabolic equations involving the Laplacian are associated explicitly with image restoration and electrorheological fluids, which are distinguished by their capacity to alter mechanical characteristics in response to external electromagnetic fields. More information on these problems is available in references [2-6]. Numerous studies have emerged concerning parabolic issues characterized by variable-exponent nonlinearities. Parabolic problems including a nonlinear exponential source term arise in several fields of applied mathematics and are used to simulate chemical processes, heat transport, and population dynamics.

This resembles a generalized reaction-diffusion equation with an exponentiated nonlinearity, often appearing in thermal conduction, porous media, reaction kinetics, heat transfer, or population dynamics. This differential equation also appears as a model in several scientific fields related to symmetry and nonlinear phenomena. Its research aids in the comprehension of complex mechanisms including biological pattern development, stellar dynamics, and diffusion.

2 Theory

This paper investigates a class of nonlinear parabolic partial differential equations (PDEs) characterized by function-exponent nonlinearities of the form [7]:

$$\frac{1}{\chi^m}\frac{d}{d\chi}\left\{\chi^m \frac{du(\chi)}{d\,\chi}\right\} + \phi^2\beta\left(\frac{1+\beta-u(\chi)}{\beta}\right)^n e^{\left(\gamma\left(\frac{u(\chi)-1}{u(\chi)}\right)\right)} = 0 \quad (1)$$

where the diffusion and reaction terms involve nonlinearities of the type $\left(\frac{1+\beta-u(\chi)}{\beta}\right)^n e^{\left(\gamma\left(\frac{u(\chi)-1}{u(\chi)}\right)\right)}$ i.e., the exponent is a nonlinear function of $u(\chi)$. The eq. (1) be a second-order nonlinear differential equation, perhaps originating from a spherical or cylindrical coordinate system according to the $\frac{1}{\chi^m}$structure (often m=0 for planar, m=1 for cylindrical, and m=2 for spherical symmetry).

The boundary conditions are

$$\text{At } \chi = 0, \frac{du}{d\chi} = 0 \quad (2)$$

$$\text{At } \chi = 1, u = 1 \quad (3)$$

By introducing the dimensionless variables

$$\chi = \frac{x}{a}, u = \frac{T}{T_b}, \gamma = \frac{E}{RT_b}, \beta = \frac{QDc_b}{KT_b}, \phi^2 = \frac{kT_b a^2 c_b^{n-1}}{D}, z_s = \frac{c}{c_b} \quad (4)$$

The effectiveness factor is

$$\eta = -\frac{(m+1)}{\Phi^2\beta}\left(\frac{du}{d\chi}\right)_{\chi=1} \quad (5)$$

Nomenclature

Symbol	Meaning	Unit
u(χ)	Dimensionless dependent variable (e.g., concentration, potential, temperature)	none
χ	Dimensionless independent variable (e.g., radial distance)	none
m	Geometrical factor (0: planar, 1: cylindrical, 2: spherical	none
Φ	Dimensionless reaction/diffusion parameter or Thiele modulus	none
β	Reaction coefficient	none
γ	Exponential reaction parameter	none
n	Power-law exponent for nonlinear kinetics	none

3. Result and discussion

3.1 Analytical expression of $u(\chi)$ and effectiveness factor using Akbari-Ganji's method (AGM).

Numerous applied problems can be resolved by solving systems of nonlinear equations. However, significant progress has been made in developing accurate approximate analytical techniques over the last three decades. Among the most popular approaches are the variation iteration method [8,9], the hyperbolic method [10,11], the AGM method [12-15], the Adomian decomposition method [16,17] and the Taylor series method [18,19,20] and HPM [21-24]and RJM [25,26]. One of the popular techniques for resolving nonlinear differential equations is AGM. Utilizing this methodology, u(χ) may be computed as follows:

The following series form is assumed to be the solution to equation (6).

$$u(\chi) = u_0 + u_1\chi + u_2\chi^2 \tag{6}$$

Where u_0, u_1, u_2 are constants. Using boundary conditions (2) and (3), we get $u_1 = 0$, $u_0 = 1 - u_2$. The solution now becomes

$$u(\chi) = 1 - u_2 + u_2\chi^2 = 1 + u_2(\chi^2 - 1) \tag{7}$$

Equation (7) is substituted in Equation (1) at χ=0, giving

$$2u_2 + 2mu_2 + \phi^2\beta\left(\frac{\beta+u_2}{\beta}\right)^n e^{\left(\gamma\left(\frac{-u_2}{1-u_2}\right)\right)} = 0 \tag{8}$$

Simplifying the above equation (7) we get

$$2u_2(1+m) + \phi^2\beta\left(\frac{\beta+u_2}{\beta}\right)^n e^{\left(\gamma\left(\frac{-u_2}{1-u_2}\right)\right)} = 0 \tag{9}$$

The numerical value of u_2 may be found using wolframalpha.com and the given parameter $(\Phi, \beta, \gamma, \mathrm{m}$ and $\mathrm{n})$values.The effectiveness factor may be calculated in this way:

$$\eta = -\frac{(m+1)}{\Phi^2\beta}\left(\frac{du}{d\chi}\right)_{\chi=1} = \frac{2(m+1)u_2}{\Phi^2\beta} \tag{10}$$

3.2 Validation of analytical result with numerical simulation

Numerical techniques are utilized to solve the nonlinear differential equation (1) .This equation is solved using the SCILAB software pdex4 function, which resolves initial-boundary value issues for PDE. Table 1-2 compare its numerical solution with the AGM method of $U(\chi)$ for various values of parameters. The maximum average error between these results is 0.8740%

Table. 1.–Validation of our analytical result (Eq. (12)) with simulation result for $u(\chi)$ for various values of Φ when $\mathrm{m} = 2, \beta = 5, \gamma = 2$, and $\mathrm{n} = 3$.

χ	$\Phi = 0.1, u_2 = -0.00843$			$\Phi = 0.5, u_2 = -0.269799$			$\Phi = 1, u_2 = -1.12012$		
	Numerical	Eq .(7) AGM	Error %	Numerical	Eq.(7) AGM	Error %	Numerical	Eq. (7) AGM	Error %
0	1.0080	1.0080	0.0000	1.2510	1.2700	1.5188	2.1600	2.1200	1.8518
0.2	1.0080	1.0080	0.0000	1.2400	1.2590	1.5322	2.1140	2.0740	1.8921
0.4	1.0070	1.0070	0.0000	1.2080	1.2260	1.4901	1.9760	1.9370	1.9737
0.6	1.0050	1.0050	0.0000	1.1550	1.1710	1.3853	1.7390	1.7090	1.7251
0.8	1.0030	1.0030	0.0000	1.0830	1.0940	1.0157	1.4000	1.3890	0.7857
1	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000
	Average error (%)		0.0000	Average error (%)		1.1570	Average error (%)		1.3714

Table. 2.–Validation of our analytical result (Eq. (12)) with simulation result for $u(\chi)$ for various values of γ when $\mathrm{m} = 1, \beta = 3, \Phi = 0.7$, and $\mathrm{n} = 3$.

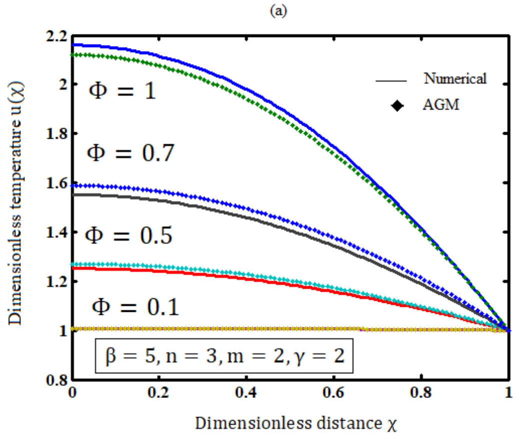
χ	$\gamma = 0.1, u_2 = -0.279971$			$\gamma = 3, u_2 = -0.580242$			$\gamma = 5, u_2 = -1.20778$		
	Numerical	Eq .(7) AGM	Error %	Numerical	Eq.(7) AGM	Error %	Numerical	Eq. (7) AGM	Error %
0	1.2970	1.2800	1.3107	1.5550	1.5800	1.6077	2.2350	2.2080	1.2080
0.2	1.2850	1.2690	1.2451	1.5320	1.5570	1.6318	2.1860	2.1580	1.2808
0.4	1.2510	1.2340	1.3589	1.4610	1.4860	1.7111	2.0350	2.0110	1.1794
0.6	1.1930	1.1770	1.3411	1.3430	1.3670	1.7870	1.7730	1.7640	0.5076
0.8	1.1080	1.0970	0.9928	1.1830	1.2010	1.5215	1.4040	1.4190	1.0684
1	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000
	Average error (%)		1.0414	Average error (%)		1.3765	Average error (%)		0.8740

3.2. The influence of parameters on $u(\chi)$.

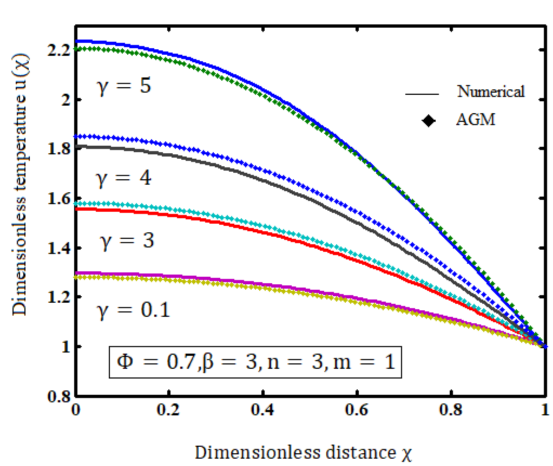
The function $u(\chi)$ depends upon the parameters ϕ, γ, β, m and n. The parameter ϕ Thiele modulus) multiplies the entire reaction term, meaning it controls the strength of the reaction relative to diffusion. The Thiele modulus changes $u(\chi)$ as shown in Figure-1(a). The curve $u(\chi)$ decreases monotonically, usually in a concave-down fashion. When ϕ is small, diffusion dominates. From the figure, it is observed that when $u(\chi)$ tends to be smoother, possibly more linear, or near-constant ($\Phi = 1$). When Φ is large, reaction dominates and rapid changes or sharp gradients in $u(\chi)$, especially near the boundaries. The impact of exponential reaction parameter γ on $u(\chi)$ is observed from the Fig-1(b).For large γ, highly sensitive exponential response occurs. This can cause the reaction term to blow up or decay sharply depending on $u(\chi)$ behavior greater1.

Figur-1(c) shows how the parameter β changes u(χ). The parameter β controls the shape of the reaction term via the shifted power law. Larger β tends to slow down the decay of the $((1+\beta-u(\chi))/\beta)^n$ term. From the fig 1(c),it is observed that u(χ) decreases when β decreases at χ=0. The impact of n (power-law exponent for nonlinear kinetics) is shown fig 1(d). It is evident from fig. 1(d) that as n rises at χ=0, u(χ) drops.

(a)



(b)



(c)

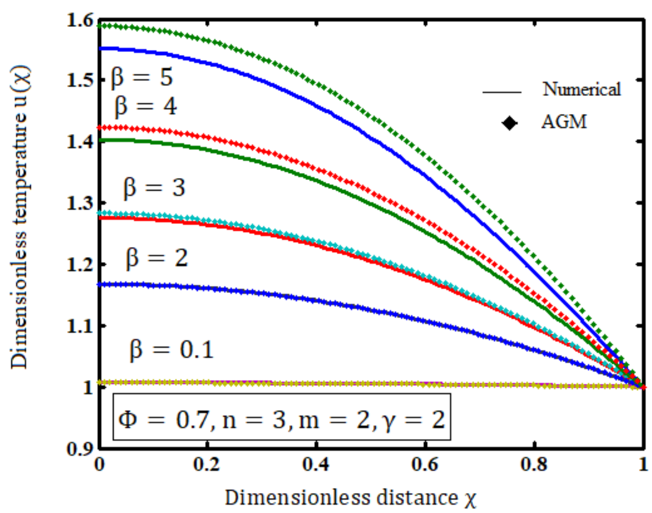


Fig.1 Effect of the parameters Φ, β, γ, n and m on dimensionless temperature $u(\chi)$ using Eq. (7).

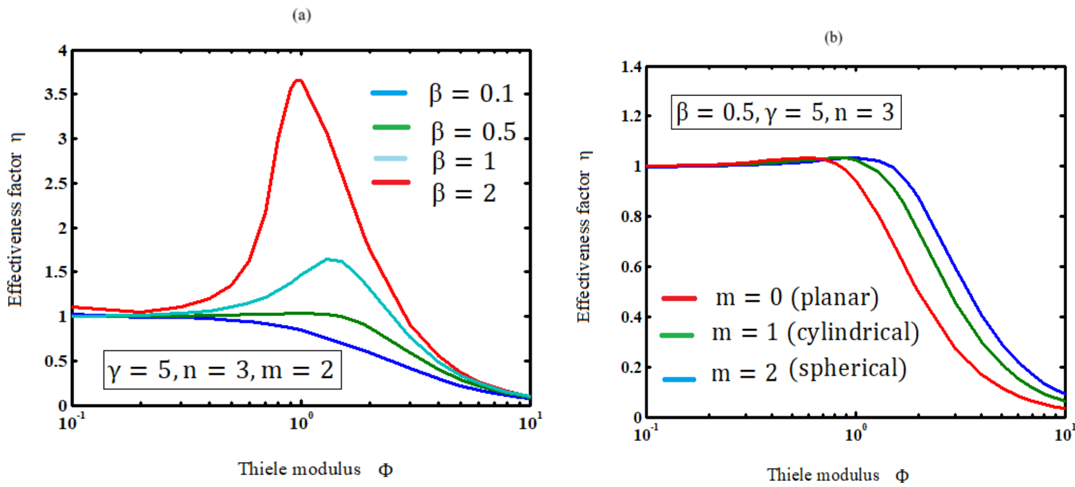


Fig.2 Plot of the effectiveness factor η versus of the Thiele modulus Φ for β and m using the Eq. (10)

The effectiveness factor is important in electrical systems, biological engineering, and catalysis because it measures how diffusion limits affect a process's efficiency. With this information, catalysts, enzymes, and reactors are designed and optimized to work at their best.The efficiency factor (η) quantifies the extent to which internal diffusion constrains the actual response rate relative to its conceptual maximum. It facilitates the optimization of catalyst or enzyme use by determining whether diffusion is constraining performance. A high η indicates effective use, while a low η denotes the need for enhancement in mass movement. Figure 2(a) shows EF as a function of Thiele modulus for various values of β . When $\phi \to 0$ diffusion is fast, so substrate is uniform and $\eta \to 1$.

When $\phi \gg 1$, diffusion limits penetration into the catalyst; most reaction occurs near the surface, in this case, $\eta \ll 1$.From the figure 2(a) , it is observed that the curve is linear when $\Phi \ll 10^{-05}$ and increase slowly an reaches the maximum at $\Phi = 1$ and decreases monotonically, usually in a concave-down fashion.

Plotting η vs ϕ shows: $\eta \sim 1$ at small φ (reaction-controlled) and η drops sharply as φ increases (diffusion-controlled. Figure 2(b) shows EF as a function of Φ for various values of β. Figure 2(b) shows that for all geometries, the lowest efficiency is at high ϕ.Figure 2(b) indicates that the efficiency factor (EF) for planar geometry is lower to that of the other two geometries, since planar geometry utilizes material more efficiently. The sphere geometry has a higher efficiency factor than the other two geometries because it has the best diffusion resistance due to its 3D shape.

Limiting case-1. When the Thiele modulus is zero, the of the u(χ) remains uniform throughout the catalyst particle due to the negligible u(χ) gradient inside it. When $\Phi^2 = 0$, the equation (12) becomes

$$\frac{1}{\chi^m}\frac{d}{d\chi}\left\{\chi^m \frac{du(\chi)}{d\,\chi}\right\}=0 \qquad (11)$$

The exact solution in this becomes $u(\chi) = 1$. Our general result equation (9) becomes $2(1-u_0)(1+m)$i.e m=0. Now equation (9) becomes $u(\chi)$= 1.

Limiting case-2. $u(\chi)$ and effectiveness factor for non-isothermal reactions.

A non-isothermal reaction transpires at varying temperatures throughout the process. This reaction is common in factories and may be used to manufacture diverse products. The nonlinear equation becomes in dimensionless form as follows:

$$\frac{1}{\chi^m}\frac{d}{d\chi}\left\{\chi^m \frac{du(\chi)}{d\,\chi}\right\} + \phi^2\beta\left(\frac{1+\beta-u(\chi)}{\beta}\right)^n e^{\left(\gamma\left(\frac{u(\chi)-1}{u(\chi)}\right)\right)} = 0 \qquad (12)$$

The dimensionless boundary conditions for non-isothermal reaction are

At $\chi = 1$, $\frac{du}{d\chi} = \sigma(1-u)$ (13)

At $\chi = 0$, $\frac{du}{d\chi} = 0$ (14)

where $u(\chi)$ is the unknown function, and σ is the Biot number(= convective mass transfer resistance/ conductive mass transfer resistance). Assume that the solution to eq. (1) is of the following simple form.

$$u(\chi) = u_0 + u_1\chi + u_2\chi^2 \qquad (16)$$

where u_0, u_1 and u_2 are constant. The values of the constant u_1 *and* u_2 are easily determined from boundary conditions (2) and (3) as $u_1 = 0$ *and* $u_2 = (\sigma - \sigma u_0 - u_0)/(\sigma+2)$. As a result, the eq. (10) becomes as follows:

$$u(\chi) = u_0 + u_2\chi^2 = u_0 + (\sigma - \sigma u_0 - u_0)/(\sigma+2)\chi^2 \qquad (17)$$

The numerical value of u_0 can be obtained as described in the section 3.The effectiveness factor is

$$\eta = \frac{(n+1)(1+\omega)}{\Phi^2}\left(\frac{du}{dx}\right)_{x=1} = \frac{2(n+1)(1+\omega)}{\Phi^2(\sigma+2)}(\sigma - \sigma u_0 - u_0) \qquad (18)$$

When $\sigma = \infty$, the above equations (17) and (18) change to equations (7) and (10), which are for isothermal conditions.

4. Conclusions

In this study, we analyzed the solutions of a parabolic differential equation with function-exponent nonlinearities using both the Akbari-Ganji method (AGM) and a Scilab-based numerical approach**.** The AGM provided an approximate analytical solution by transforming the nonlinear equation into a solvable form while minimizing residual errors. The numerical solution, obtained using scilab served as a benchmark to validate the accuracy of the AGM approximation. A comparison of both approaches highlights that AGM is well-suited for obtaining quick approximate solutions, while numerical methods offer higher accuracy, particularly for strongly nonlinear problems. Future work can focus on extending AGM for more complex boundary conditions and exploring other numerical schemes, such as finite element **or** spectral methods**,** to enhance computational efficiency**.**

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the research reported in this paper.

Data availability

No data was used for the research described in the article

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