Analysis of Linear Equations in Nitrifying Trickling Filters Model Using Taylor Series Method

A. Marimuthu[a], P. Jeyabarathi[b], L. Rajendran[a,*]
[a] Department of Mathematics, AMET University, Chennai-603112,India
[b] Department of Mathematics, Bharath Institute of Higher Education and Research, Chennai, India.
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Abstract

The mathematical model for nitrifying trickling filters is analyzed. Tricking filters are biofilm reactors commonly used for the biological removal of nitrogen and organic matter. These models are based on the system of linear equations. The ammonium and nitrate concentrations are derived by solving linear equations using the Taylor series method.

Keywords: Mathematical modeling, Linear equations, Nitrification, Trickling filters, Taylor series method.

1.Introduction

Trickling filters have been used for over a hundred years and are among the most frequent wastewater treatment procedures. A trickling filter is a biofilm substratum consisting of a plastic or mineral inert medium linked to a growing bioreactor. Water trickles down a tower of packed media, and the microorganisms in the biofilm degrade organic matter, nitrify, denitrify, and so on, depending on the operating conditions [1]. Slow and fast modes exist in biofilm reactors, such as nitrifying trickling filters. The reactor hydraulics and diffusive mass transfer in the biofilm are primarily responsible for the fast dynamics. In contrast, the growth and decay of the organisms in the biofilm are responsible for the slow dynamics.

In nitrifying trickling filters, the fast and slow models are separated in time, so the slow transients can be ignored when only the fast dynamics are studied [2]. However, if the substrate load varies quickly, fast dynamics play an essential role in reactor efficiency [3]. A physically-based model presents the fast dynamics of nitrifying trickling filters with planar biofilm substrate. The model is based on dividing the filter into series of continuously stirred biofilm reactors, which is a method that has primarily been used in studies of slow dynamics and steady-state behaviour by [4-9]. Wik and Breitholtz [10] have indicated how to obtain rational approximations of transfer functions for continuously stirred biofilm reactors with planar, cylindrical or spherical substrata when only one substrate is assumed. In this paper, to find a consistent approximate solution, a Taylor series technique [11] is used to solve non-linear equations of concentration of substrate ammonium and nitrate for all values of parameters.

2. Mathematical formulation of the problem.

The reaction treated is the complete nitrification of ammonium into nitrate [11]:

$$NH_4^+ + 2HCO_3^- + 2O_2 \rightarrow NO_3^- + 3H_2O + 2CO_2 \qquad (1)$$

where only changes in ammonium and nitrate concentrations are considered. The substrate concentrations S in the biofilm are continuous in time (t) and space (x). The transport of substrates inside the biofilm obeys Fick's law of diffusion in one dimension, and the time scale considered is too short to cause any changes in the bacterial concentration. The dimensional mass balances over the bulk volume is continuously stirred biofilm reactors (CSBR) are

$$\tau \frac{dS_{NH_4}^b}{d\bar{t}} = S_{NH_4,in}^b - S_{NH_4}^b - \gamma_1 \frac{\partial S_{NH_4}}{\partial \xi}\bigg|_{\xi=1} \qquad (2)$$

$$\tau \frac{dS_{NO_3}^b}{d\bar{t}} = S_{NO_3,in}^b - S_{NO_3}^b - \gamma_2 \frac{\partial S_{NO_3}}{\partial \xi}\bigg|_{\xi=1} \qquad (3)$$

The parameters are

$$\bar{t} = \frac{tD_{NH_4}}{L^2(\varepsilon+k_a))}, \xi = \frac{x}{L}, \tau = \frac{VD_{NH_4}}{(\varepsilon+k_a)L^2Q}, \gamma_1 = \frac{AD_{NH_4}}{QL} \text{and } \gamma_2 = \frac{AD_{NO_3}}{QL} \qquad (4)$$

Where $\bar{t}$ is a time and space, τ is a time-scaling coefficient, V is a bulk water volume and Q volumetric flow rate, A is a area of biofilm and ξ is a scaled distance from origin (x/L). γ_1 and γ_2 is the non-dimensional coefficient for substrate flux into biofilm. Assuming a linear dependence of the intrinsic reaction rate r_v on the ammonium concentration,

$$r_{v,NH_4} = k_2 + k_1 S_{NH_4}$$
(5)

The dimensionless mass balances in the biofilm becomes

$$\frac{dS_{NH_4}}{d\bar{t}} = \frac{\partial^2 S_{NH_4}}{\partial \xi^2} - k_1 S_{NH_4} - \mu_1 \qquad (6)$$

$$\eta \frac{dS_{NO_3}}{d\bar{t}} = \frac{\partial^2 S_{NO_3}}{\partial \xi^2} + k_2 S_{NH_4} + \mu_2 \qquad (7)$$

The steady-steady condition for dimensionless substrate concentrations of ammonium and nitrate becomes,

$$\frac{d^2 S_{NH_4}}{d\xi^2} - \kappa_1 S_{NH_4} - \mu_1 = 0 \tag{8}$$

$$\frac{d^2 S_{NO_3}}{d\xi^2} + \kappa_2 S_{NH_4} + \mu_2 = 0 \tag{9}$$

The boundary condition becomes,

$$\text{At } \xi = 0, \frac{dS_{NH_4}}{d\xi} = 0 \text{and} \frac{dS_{NO_3}}{d\xi} = 0 \tag{10}$$

$$\text{At } \xi = 1, S_{NH_4} = S^b_{NH_4} \text{ and } \quad S_{NO_3} = S^b_{NO_3} \tag{11}$$

The dimensionless parameters are

$$\kappa_1 = \frac{L^2 k_1}{D_{NH_4}}, \mu_1 = \frac{L^2 k_2}{D_{NH_4}}, \eta = \frac{D_{NH_4}\varepsilon}{D_{NO_3}(\varepsilon + k_a)}, \kappa_2 = \frac{L^2 k_1}{D_{NO_3}} \text{and } \mu_2 = \frac{L^2 k_2}{D_{NO_3}} \tag{12}$$

Where S_{NH_4} and S_{NO_3} are substrate concentrations of ammonium and nitrate, κ_1 is the dimensionless first order rate coefficient and κ_2 are the dimensionless zero order rate coefficient, μ_1 and μ_2 are the specific growth rate, L is a biofilm thickness, D_{NH_4} and D_{NO_3} are the diffusion coefficients of ammonium and nitrate, ε is a biofilm porosity, η are the time constant, k_1 are the zero order rate constant , k_2 are the first order rate constant and k_a is a adsorption coefficient .The schematic diagram of continuously stirred biofilm reactors is given in Fig-1.

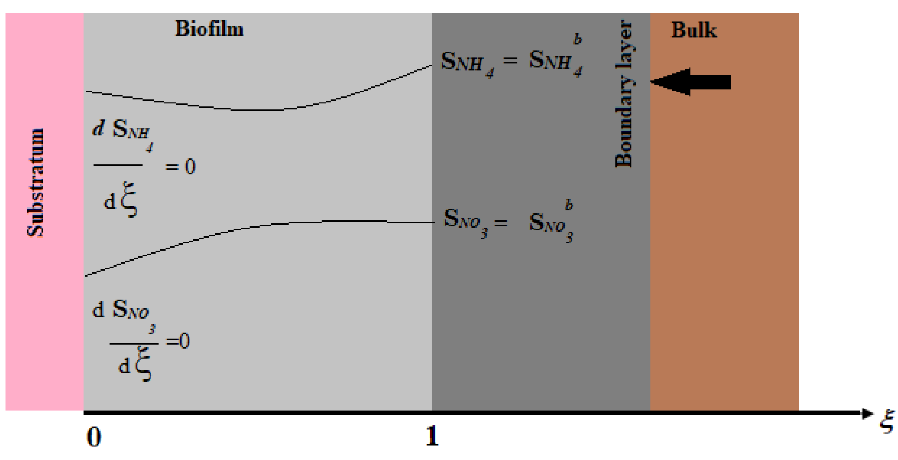

Fig. 1. Schematic diagram of continuously stirred biofilm reactors.

3. Analytical expressions of the concentrations of ammonium and nitrate using Taylor series method.

Numerical solutions of the ordinary differential equations (ODEs) by using the Taylor series method have been investigated by many authors [12–18] and references there in. However, there are few references on the solution of the differential equations by using the Taylor series method. One advantage of using the Taylor series is that a differentiable approximate solution is obtained, which can be replaced with the equation and the initial or boundary conditions. In this manner, the accuracy of the solution can be evaluated directly. Since this method is necessary to compute the partial derivatives of the right side of the equation, a series of algorithms were found to approximate the partial results [16].The dimensionless substrate concentrations of ammonium and nitrate can be obtained by solving the linear equations (8) and (9) using Taylors series method as follows:

$$S_{NH_4}(\xi) = \sum_{i=0}^{6} \frac{d^i S_{NH_4}}{d\xi^i}\bigg|_{\xi=0} \frac{\xi^i}{i!} = S_{NH_4}(0) + \frac{dS_{NH_4}}{d\xi}\bigg|_{\xi=0} \frac{\xi}{1!} + \frac{d^2 S_{NH_4}}{d\xi^2}\bigg|_{\xi=0} \frac{\xi^2}{2!} + \frac{d^3 S_{NH_4}}{d\xi^3}\bigg|_{\xi=0} \frac{\xi^3}{3!} + .. \quad (13)$$

$$S_{NO_3}(\xi) = \sum_{i=0}^{6} \frac{d^i S_{NO_3}}{d\xi^i}\bigg|_{\xi=0} \frac{\xi^i}{i!} = S_{NO_3}(0) + \frac{dS_{NO_3}}{d\xi}\bigg|_{\xi=0} \frac{\xi}{1!} + \frac{d^2 S_{NO_3}}{d\xi^2}\bigg|_{\xi=0} \frac{\xi^2}{2!} + \frac{d^3 S_{NO_3}}{d\xi^3}\bigg|_{\xi=0} \frac{\xi^3}{3!} + .. \quad (14)$$

By differentiating the above Eq. (8) successively with respect to 'ξ', we get the following results.

$$\frac{d^2 S_{NH_4}}{d\xi^2}\bigg|_{\xi=0} = \kappa_1 S_{NH_4}(0) + \mu_1,$$

$$\frac{d^3 S_{NH_4}}{d\xi^3}\bigg|_{\xi=0} = 0,$$

$$\frac{d^4 S_{NH_4}}{d\xi^4}\bigg|_{\xi=0} = \kappa_1^2 S_{NH_4}(0) + \kappa_1 \mu_1$$

$$\frac{d^5 S_{NH_4}}{d\xi^5}\bigg|_{\xi=0} = 0$$

$$\frac{d^6 S_{NH_4}}{d\xi^6}\bigg|_{\xi=0} = \kappa_1^3 S_{NH_4}(0) + \kappa_1^2 \mu_1 \quad (15)$$

Using Eq. (13) in Eq. (15), the dimensionless substrate concentrations of ammonium

$$S_{NH_4}(\xi) = S_{NH_4}(0) + (\kappa_1 S_{NH_4}(0) + \mu_1)\frac{\xi^2}{2!} + (\kappa_1^2 S_{NH_4}(0) + \kappa_1\mu_1)\frac{\xi^4}{4!} + (\kappa_1^3 S_{NH_4}(0) + \kappa_1^2\mu_1)\frac{\xi^6}{6!} \quad (16)$$

Using the boundary conditions $\xi = 1,\ S_{NH_4}(\xi) = S_{NH_4}^b$ in Eq. (16) implies that

$$S_{NH_4}^b = S_{NH_4}(0) + (\kappa_1 S_{NH_4}(0) + \mu_1)\frac{1}{2!} + (\kappa_1^2 S_{NH_4}(0) + \kappa_1\mu_1)\frac{1}{4!} + (\kappa_1^3 S_{NH_4}(0) + \kappa_1^2\mu_1)\frac{1}{6!} \quad (17)$$

The value of $S_{NH_4}(0)$ can be obtained by solving the above Eq. (17).

Similarly, by solving the eq. (9) we can obtainthe concentration of nitrate as follows:

$$S_{NO_3}(\xi) = S_{NO_3}(0) - (\kappa_2 S_{NH_4}(0) + \mu_2)\frac{\xi^2}{2!} - \kappa_2(\kappa_1 S_{NH_4}(0) + \mu_1)\frac{\xi^4}{4!} - \kappa_2(\kappa_1^3 S_{NH_4}(0) + \kappa_1^2\mu_1)\frac{\xi^6}{6!} \quad (18)$$

Applying the Eq. (10) in Eq. (18) implies that

$$S_{NO_3}^b = S_{NO_3}(0) - (\kappa_2 S_{NH_4}(0) + \mu_2)\frac{1}{2!} - \kappa_2(\kappa_1 S_{NH_4}(0) + \mu_1)\frac{1}{4!} - \kappa_2(\kappa_1^3 S_{NH_4}(0) + \kappa_1^2\mu_1)\frac{1}{6!} \quad (19)$$

The value of $S_{NO_3}(0)$ can be obtained by solving the above Eq. (19).

4. Exact solution result

The substrate concentration of ammonium and nitrate are

$$S_{NH_4} = \left(S_{NH_4}^b + \frac{\mu_1}{\kappa_1}\right)\frac{cosh(\sqrt{\kappa_1}x)}{cosh(\sqrt{\kappa_1})} - \frac{\mu_1}{\kappa_1} \quad (20)$$

$$S_{NO_3} = \frac{\kappa_2}{\kappa_1}(S_{NH_4}^b + \mu_1)\left(1 - \frac{cosh(\sqrt{\kappa_1}x)}{cosh(\sqrt{\kappa_1})}\right) + \frac{1}{2}\left(\frac{\kappa_2\mu_1}{\kappa_1} - \mu_2\right)(x^2 - 1) + S_{NO_3}^b \quad (21)$$

5. Discussion

Equations (16) and (18) are the new closed and simple approximate analytical expressions of substrate concentration of ammonium and nitrate for all parameter values.

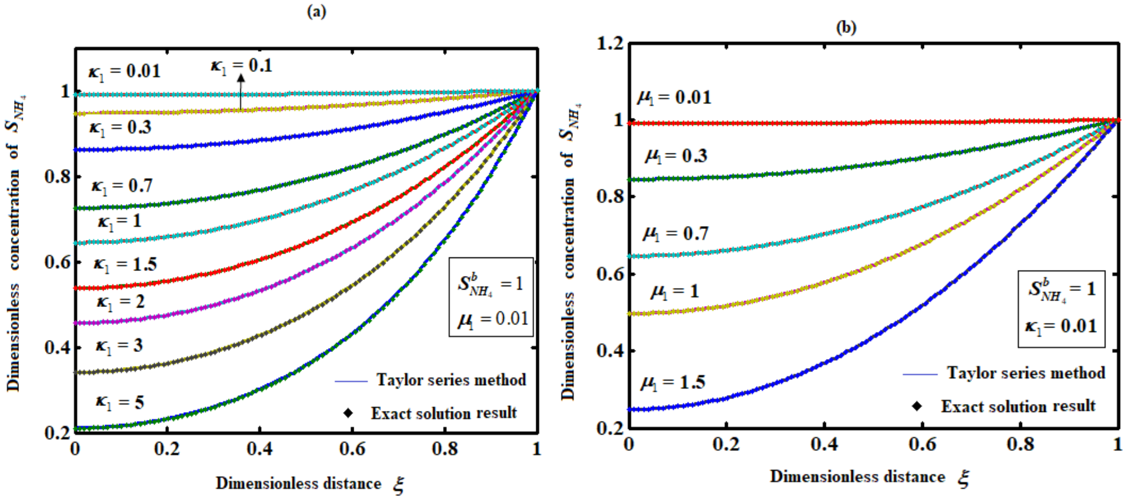

Fig. 2 Dimensionless concentration of S_{NH_4} versus dimensionless distance ξ for various values of parameters μ_1 and κ_1 using Eq. (16) in Taylor series result and Eq. (20) in exact solution result.

The ammonium substrate concentration profiles are shown in Fig. 2 a-b. As the standardized parameter κ_1 decreases, substrate concentration of ammonium decreases. The substrate concentration of ammonium decreases, as the parameter μ_1 decreases.

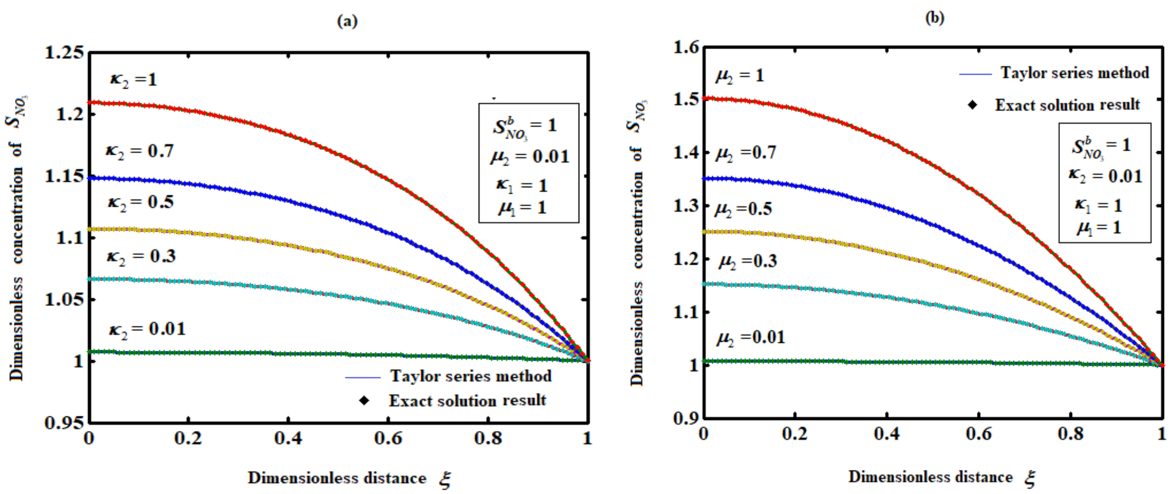

Fig. 3 Dimensionless concentration of S_{NO_3} versus dimensionless distance ξ for various values of parameters μ_2 and κ_2 using Eq. (18) in Taylor series result and Eq. (21) in exact solution result.

From Fig. 3b, it is inferred that the concentration of nitrate increases as the parameter μ_2 increases. The substrate concentration of nitrate increases as the parameters κ_2 increases.

6. Conclusion

In the present work, the Taylor series is used to solve the second order linear differential equations in Tricking filters. The solution of differential equations has been reduced to a problem of solving a system of algebraic equations. The method can be extended to more independent variables partial differential equations. The method used only algebraic methods to approximate the solutions of ODE/PDE problems and it has an efficient method for error reduction. The approximate analytical expression of concentrations of ammonium and nitrate for all experimental values of parameters is derived using the Taylor series method compared with the exact solution result. These analytical expressions can be used to analyze the effect of zero order rate coefficient, non dimensionless zero order rate coefficient moment specific growth rate.

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Nomenclature

Symbols	Meanings	Units
A	Biofilm area	m^2
D_{NH_4}	Diffusion coefficient of ammonium	$m^2\,d^{-1}$
D_{NO_3}	Diffusion coefficient of nitrate	$m^2\,d^{-1}$
k_1	Zero order rate coefficient	$gm^{-3}d^{-1}$
k_2	First order rate coefficient	d^{-1}
k_a	Adsorption coefficient	$gm^{-3}(gm^{-3})^{-1}$
L	Biofilm thickness	m
Q	Volumetric flow rate	$m^3\,d^{-1}$
r_v	Specific substrate production rate	$gm^{-3}d^{-1}, mole\,m^{-3}d^{-1}$
S_{NH_4}	Substrate concentration of ammonium	-
S_{NO_3}	Substrate concentration of nitrate	-
$S_{NH_4}^b, S_{NO_3}^b$	Substrate concentration in the bulk	-
V	Bulk water volume	m^3
t	Time	d
x	Distance from substratum	m
γ	Non dimensional coefficient for substrate flux into biofilm	-
ε	Biofilm porosity	m^3m^{-3}
κ	Non dimensionless first order rate coefficient	-
μ	Non dimensionless zero order rate coefficient moment specific growth rate	d^{-1}
τ	Time constant	d

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