Generalized Fixed-Point Theorems in Bn-Metric Spaces and Their Applications

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ABSTRACT. In this paper, we present a general fixed-point theorem for B_n-metric spaces, which generalizes fixed-point results from S-metric spaces. As applications, we derive several analogues of well-known fixed-point theorems in metric spaces, extending them to B_n-metric spaces.

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1. Introduction and Preliminaries

The generalizations of the contraction principle in different directions as well as many new fixed-point results with applications have been established by different researchers ([**1**], [**2**], [**7**], [**11**], [**13**], [**15**], [**17**]).

In [**16**], Ch. Srinivasa Rao, S. Ravi Kumar, and K.K.M. Sarma introduced the notion of a B_n-metric space as follows.

DEFINITION 1.1. Let X be a non-empty set. A B_n-metric on X is a function $d : X \times X \to \mathbb{R}$ that satisfies the following conditions for all $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$ (symmetry).
(iii) $d(x, z) \leqslant b[d(x, y) + d(y, z)]$, for some $b \geqslant 1$ (generalized triangle inequality).

The pair (X, d) is called a B_n-metric space.

This notion generalizes both b-metric spaces and S-metric spaces [**14**]. For the fixed-point problem in generalized metric spaces, numerous results have been proved. See, for example, [**1**]. [**7**], [**9**], [**10**], for example in [**16**], the authors proved some properties of B_n-metric spaces. Also, they proved Some fixed-point theorems for a self-map on an B_n-metric space.

Now we recall some notions and lemmas that will be useful later.

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Definition 1.2. , [**16**]] Let X be a nonempty set. A B_n-metric on X is a function $B_n : X^n \to [0,\infty)$ such that there exists a real number $\lambda \geqslant 1$ satisfying the following conditions for all $x_1, x_2, \ldots, x_n, a \in X$:

(B1) $B_n(x_1,x_2,\ldots,x_n) = 0$ if and only if $x_1 = x_2 = \cdots = x_n$.
(B2) $B_n(x_1,x_2,\ldots,x_n) = B_n(x_n,x_{n-1},\ldots,x_1)$.
(B3) $B_n(x_1,x_2,\ldots,x_n) \leqslant \lambda\Big[B_n(x_1,x_1,\ldots,x_1,a)+B_n(x_2,x_2,\ldots,x_2,a)+\cdots+B_n(x_n,x_n,\ldots,x_n,a)\Big]$.

The pair (X, B_n) is called a B_n-metric space.

Definition 1.3. , [**16**]] Let (X, B_n) be a B_n-metric space. For $r > 0$, the open ball $B_S(x,r)$ and the closed ball $B_S[x,r]$ with center x and radius r are defined as follows:

$$B_S(x,r) = \{y \in X : B_n(y,y,y,x) < r\},$$
$$B_S[x,r] = \{y \in X : B_n(y,y,y,x) \leqslant r\}.$$

The topology induced by the B_n-metric is the topology generated by the base of all open balls in X.

Definition 1.4. , [**16**] Let (X, B_n) be a B_n-metric space.

(1) A sequence $\{x_n\} \subset X$ converges to $x \in X$ if $B_n(x_n,x_n,\ldots,x_n,x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, we have $B_n(x_n,x_n,\ldots,x_n,x) < \varepsilon$. We write $x_n \to x$ for brevity.
(2) A sequence $\{x_n\} \subset X$ is a Cauchy sequence if $B_n(x_n,x_n,\ldots,x_n,x_m) \to 0$ as $n,m \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n,m \geqslant n_0$, we have $B_n(x_n,x_n,\ldots,x_n,x_m) < \varepsilon$.
(3) The B_n-metric space (X, B_n) is complete if every Cauchy sequence is convergent.

Lemma 1.1. , [**16**] *In a B_n-metric space, we have:*

$$B_n(x_1,x_1,\ldots,x_1,x_2) = B_n(x_2,x_2,\ldots,x_2,x_1), \quad \textit{for all } x_1,x_2 \in X.$$

Lemma 1.2. , [**16**] *Let (X, B_n) be a B_n-metric space. If $x_{1_n} \to x_1$, $x_{2_n} \to x_2$, $\ldots$, $x_{n-1_n} \to x_{n-1}$, then:*

$$B_n(x_{1_n},x_{1_n},x_{2_n},\ldots,x_{n-1_n}) \to B_n(x_1,x_1,x_2,\ldots,x_{n-1}).$$

Example 1.1. Let $\mathbb{R}$ be the real line. Define:

$$B_n(x_1,x_2,\ldots,x_n) = |x_1 - x_n| + |x_2 - x_n| + \cdots + |x_1 + x_2 + \cdots + x_{n-1} - (n-1)x_n|,$$

for all $x_1,x_2,\ldots,x_n \in \mathbb{R}$. Then B_n is a B_n-metric on $\mathbb{R}$. This B_n-metric is called the usual B_n-metric on $\mathbb{R}$.

2. Main Results

First, we prove some properties of B_n-metric spaces.

Proposition 2.1. *Let (X, B_n) be a B_n-metric space, and let*

$$B_{n-1}(x_1,x_2,\ldots,x_{n-1}) = B_n(x_1,x_1,x_2,\ldots,x_{n-1})+B_n(x_1,x_2,x_2,\ldots,x_{n-1})+\cdots+B_n(x_1,x_2,\ldots,x_{n-1},x_{n-1}),$$

for all $x_1,x_2,\ldots,x_{n-1} \in X$. Then we have the following:

(1) *B_{n-1} is a B_n-metric on X.*
(2) *$x_n \to x$ in (X, B_n) if and only if $x_n \to x$ in (X, B_{n-1}).*
(3) *$\{x_n\}$ is a Cauchy sequence in (X, B_n) if and only if $\{x_n\}$ is a Cauchy sequence in (X, B_{n-1}).*

Proposition 2.2. *Let (X, B_n) be a B_n-metric space. Then:*

(1) *X is first-countable.*
(2) *X is regular.*

Remark 2.1. By Propositions 2.1 and 2.2, we have that every B_n-metric space is topologically equivalent to a B_n-metric space.

Corollary 2.1. *Let $f : X \to Y$ be a map from a B_n-metric space X to a B_n-metric space Y. Then f is continuous at $x \in X$ if and only if $f(x_n) \to f(x)$ whenever $x_n \to x$.*

theorem 2.1. *Let T be a self-map on a complete B_n-metric space (X, B_n), and let*

$$B_n(Tx_1, Tx_1, \dots, Tx_1, Tx_2) \leqslant M\big(B_n(x_1, x_1, \dots, x_1, x_2), B_n(Tx_1, Tx_1, \dots, Tx_1, x_1),$$
$$B_n(Tx_1, Tx_1, \dots, Tx_1, x_2), B_n(Tx_2, Tx_2, \dots, Tx_2, x_1), B_n(Tx_2, Tx_2, \dots, Tx_2, x_2)\big),$$

for all $x_1, x_2, \dots, x_n \in X$ and some $M \in \mathcal{M}$. Then:

(1) *If M satisfies condition (C1), then T has a fixed-point. Moreover, for any $x_0 \in X$ and the fixed-point x, we have*

$$B_n(Tx_n, Tx_n, \dots, Tx_n, x_n) \leqslant \frac{1-k}{2k^n} B(x_0, x_0, \dots, x_0, Tx_0).$$

(2) *If M satisfies condition (C2) and T has a fixed-point, then the fixed-point is unique.*
(3) *If M satisfies condition (C3) and T has a fixed-point x, then T is continuous at x.*

Corollary 2.2. , [**12**] *Let T be a self-map on a complete B_n-metric space (X, B_n) and*

$$B_n(Tx, Tx, \dots, Tx, Ty) \leqslant LB_n(x, x, \dots, x, y),$$

for some $L \in [0, 1)$ and all $x, y \in X$. Then T has a unique fixed-point in X. Moreover, T is continuous at the fixed-point.

Proof. The assertion follows using Theorem 2.6 with:

$$M(x_1, x_2, \dots, x_{n+1}) = Lx,$$

for some $L \in [0, 1)$ and all $x_1, x_2, \dots, x_{n+1} \in \mathbb{R}_+$. □

Corollary 2.3. *Let T be a self-map on a complete B_n-metric space (X, B_n), and suppose:*

$$B_n(Tx, Tx, \dots, Tx, Ty) \leqslant a\big(B_n(Tx, Tx, \dots, Tx, x) + B_n(Ty, Ty, \dots, Ty, y)\big),$$

for some $a \in [0, 1/2)$ and all $x, y \in X$. Then T has a unique fixed-point in X. Moreover, T is continuous at the fixed-point.

Proof. The assertion follows using Theorem 2.6 with:

$$M(x_1, x_2, \dots, x_{n+1}) = a(x_2 + x_{n+1}),$$

for some $a \in [0, 1/2)$ and all $x_1, x_2, \dots, x_{n+1} \in \mathbb{R}_+$.
Indeed, M is continuous.
Condition (C1): We have:

$$M(x_1, x_1, \dots, 0, \dots, x_n, x_{n-1}) = a(x_1 + \cdots + x_{n-1}).$$

If $y \leqslant M(x_1, x_1, 0, x_{n+1}, x_n)$ with $x_{n+1} \leqslant 2x_1 + x_n$, then:

$$y \leqslant \frac{a}{1-a} x, \quad \text{where } \frac{a}{1-a} < 1.$$

Therefore, T satisfies condition (C1).
Condition (C2): If:

$$x_n \leqslant M(x_n, 0, \dots, x_n, x_n, 0, \dots) = 0,$$

then $x_n = 0$. Thus, T satisfies condition (C2).
Condition (C3): If $x_i \leqslant y_i + z_i$ for $i \leqslant n + 1$, then:

$$M(x_1, \dots, x_{n+1}) = a(x_2 + \cdots + x_{n+1}) \leqslant a\big[(y_2 + z_2) + \cdots + (y_{n+1} + z_{n+1})\big],$$
$$= a(y_2 + z_2) + \cdots + a(y_{n+1} + z_{n+1}) = M(y_1, \dots, y_{n+1}) + M(z_1, \dots, z_{n+1}).$$

Moreover:

$$M(0, 0, \dots, 0, y, 2y) = a(0 + 2y) = 2ay, \quad \text{where } 2a < 1.$$

Therefore, T satisfies condition (C3). □

REMARK 2.2. Note that the coefficient k in conditions (C1) and (C3) may be different, for example, k_1 and k_3 respectively. However, we may assume that they are equal by setting $k = \max\{k_1, k_3\}$.

EXAMPLE 2.1. Let $\mathbb{R}$ be the usual B_n-metric space as in Example 1.7, and define T by:

$$T(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in [0,1), \\ \frac{1}{4}, & \text{if } x = 1. \end{cases}$$

Then T is a self-map on a complete B_n-metric space $[0,1] \subset \mathbb{R}$.

For all $x \in (3/4, 1)$, we have:

$$B_n(Tx, Tx, \ldots, Tx) = B_n\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{4}\right) = \left|\frac{1}{2} - \frac{1}{4}\right| + \cdots + \left|\frac{1}{2} - \frac{1}{4}\right| = \frac{1}{2},$$

and

$$B_n(x, x, \ldots, 1) = |x-1| + \cdots + |x-1| = n|x-1| < \frac{1}{2}.$$

Thus, T does not satisfy the condition of Corollary 2.7.

Next, we have:

$$B_4(Tx, Tx, Tx, Ty) \leqslant \frac{5}{12}\left(B_4(Tx, Tx, Tx, x) + B_4(Ty, Ty, Ty, y)\right),$$

which shows that T satisfies the condition of Corollary 2.8. It is clear that $x = \frac{1}{2}$ is the unique fixed-point of T.

COROLLARY 2.4. *Let T be a self-map on a complete B_n-metric space (X, B_n), and suppose:*

$$B_n(Tx, Tx, \ldots, Tx, Ty) \leqslant h\left(B_n(Tx, Tx, \ldots, Tx, x) + B_n(Ty, Ty, \ldots, Ty, y)\right),$$

for some $h \in [0,1)$ and all $x, y \in X$. Then T has a unique fixed-point in X. Moreover, if $h \in [0, 1/2)$, then T is continuous at the fixed-point.

PROOF. The assertion follows using Theorem 2.6 with:

$$M(x_1, x_2, \ldots, x_{n+1}) = h \max(x_{n-2}, x_{n+1}),$$

for some $h \in [0,1)$ and all $x_1, \ldots, x_{n+1} \in \mathbb{R}_+$.

Indeed, M is continuous. First, we have:

$$M(x_1, x_1, \ldots, 0, \ldots, x_n, x_{n-1}) = h \max(x, y).$$

If $y \leqslant M(x_1, x_1, 0, \ldots, x_n, x_{n-1})$ with $x_n \leqslant 2x_1 + x_{n-1}$, then:

$$y \leqslant hx \quad \text{or} \quad x_{n-1} \leqslant hx_{n-1}.$$

Therefore, $x_{n-1} \leqslant hx_1$, and hence, T satisfies condition (C1).

Next, if $y \leqslant M(y, 0, y, y, 0) = h\max(y, 0) = hy$, then $y = 0$ since $h < 1/2$. Therefore, T satisfies condition (C2).

Finally, if $x_i \leqslant y_i + z_i$ for $i \leqslant n+1$, then:

$$M(x_1, \ldots, x_{n+1}) = h(x_2, x_{n+1}) \leqslant h\left(y_2 + z_2, y_{n+1} + z_{n+1}\right) \leqslant h\left(y_2, y_{n+1}\right) + h\left(z_2, z_{n+1}\right)$$

$$= M(y_1, \ldots, y_{n+1}) + M(z_1, \ldots, z_{n+1}).$$

Moreover, if $h \in [0, 1/2)$, then $2h < 1$ and:

$$M(0, 0, \ldots, 0, y, 2y) = h(0 + 2y) = 2hy,$$

where $2h < 1$.

Therefore, T satisfies condition (C3). □

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EXAMPLE 2.2. Let $\mathbb{R}$ be the usual B_n-metric space as in Example 1.7, and define $T(x) = \frac{x}{n}$ for all $x \in [0,1]$. We have:

$$B_n(Tx, Tx, \dots, Tx, Ty) = B_n\left(\frac{x}{n}, \frac{x}{n}, \dots, \frac{x}{n}, \frac{y}{n}\right) = \left|\frac{x}{n} - \frac{y}{n}\right| + \left|\frac{x}{n} - \frac{y}{n}\right| + \cdots + \left|\frac{x}{n} - \frac{y}{n}\right|$$

$$= \frac{n-1}{n}|x-y|.$$

Next, for the case where $x = T(x)$:

$$B_n(Tx, Tx, \dots, Tx, x) = B_n\left(\frac{x}{n}, \frac{x}{n}, \dots, \frac{x}{n}, x\right) = \left|\frac{x}{n} - x\right| + \left|\frac{x}{n} - x\right| + \cdots$$

$$= \frac{n}{n}|x| = |x|.$$

Similarly, for $y = T(y)$:

$$B_n(Ty, Ty, \dots, Ty, y) = B_n\left(\frac{y}{n}, \frac{y}{n}, \dots, \frac{y}{n}, y\right) = \frac{n}{n}|y| = |y|.$$

Thus, for all $x, y \in X$:

$$B_n(Tx, Tx, \dots, Tx, x) + B_n(Ty, Ty, \dots, Ty, y) = \frac{n}{n}(|x| + |y|).$$

Finally, we have:

$$B_n(Tx, Tx, \dots, Tx, Ty) = \frac{n-1}{n} B_n(Tx, Tx, \dots, Tx, 1) + B_n(T0, T0, \dots, T0, 0) = 1.$$

This proves that T does not satisfy the condition of Corollary 2.8. However, we do have that T satisfies the condition of Corollary 2.10 with $h = \frac{n-1}{n}$, and T has a unique fixed-point at $x = 0$.

COROLLARY 2.5. *Let T be a self-map on a complete B_n-metric space (X, B_n), and suppose:*

$$B_n(Tx, Tx, \dots, Tx, Ty) \leqslant aB_n(x, x, \dots, x, y) + bB_n(Tx, Tx, \dots, Tx, x) + cB_n(Ty, Ty, \dots, Ty, y),$$

for some $a, b, c \geqslant 0$, and $a + b + c < 1$, and for all $x, y \in X$. Then T has a unique fixed-point in X. Moreover, if $c < 1/2$, then T is continuous at the fixed-point.

PROOF. The assertion follows using Theorem 2.6 with:

$$M(x_1, x_2, \dots, x_{n+1}) = ax_1 + bx_2 + \cdots + cx_{n+1},$$

for some $a, b, c \geqslant 0$, with $a + b + c < 1$ and all $x_1, \dots, x_{n+1} \in \mathbb{R}_+$. Indeed, M is continuous.

First, we have:

$$M(x_1, x_1, \dots, 0, x_n, x_{n-1}) = ax_1 + bx_2 + \cdots + cx_{n+1}.$$

If $y \leqslant M(x_1, x_1, \dots, 0, x_n, x_{n-1})$ with $x_n \leqslant 2x_{n-2} + x_{n-1}$, then:

$$x_{n-1} \leqslant \frac{a+b}{1-c} x_{n-2}, \quad \text{where} \quad \frac{a+b}{1-c} < 1.$$

Therefore, T satisfies condition (C1).

Next, if $x_{n-1} \leqslant M(x_{n-1}, \dots, 0, \dots, x_{n-1}, x_{n-1}, \dots) = ax_{n-1}$, then $x_{n-1} = 0$, since $a < 1$. Therefore, T satisfies condition (C2).

Finally, if $x_i \leqslant y_i + z_i$ for $i \leqslant n+1$, then:

$$M(x_1, \dots, x_{n+1}) = ax_1 + bx_2 + \cdots + cx_{n+1} \leqslant a(y_1 + z_1) + b(y_2 + z_2) + \cdots + c(y_{n+1} + z_{n+1})$$

$$= (ay_1 + by_2 + \cdots + cy_{n+1}) + (az_1 + bz_2 + \cdots + cz_{n+1}) = M(y_1, \dots, y_{n+1}) + M(z_1, \dots, z_{n+1}).$$

Moreover:

$$M(0, 0, \dots, 0, x_{n-1}, 2x_{n-1}) = 2cx_{n-1}, \quad \text{where} \quad 2c < 1.$$

Therefore, T satisfies condition (C3). □

Example 2.3. Let $\mathbb{R}$ be the usual B_n-metric space as in Example 1.7, and define $T(x) = \frac{x}{n-2}$ for all $x \in [0,1]$. We have:

$$B_n(Tx, Tx, \ldots, Tx, Ty) = \left|\frac{x}{n-2} - \frac{y}{n-2}\right| + \left|\frac{x}{n-2} - \frac{y}{n-2}\right| + \cdots + \left|\frac{x}{n-2} - \frac{y}{n-2}\right|$$

$$= \frac{n-1}{n-2}|x-y|.$$

Next, for the case where $x = T(x)$:

$$B_n(Tx, Tx, \ldots, Tx, x) = \left|\frac{x}{n-2} - x\right| + \left|\frac{x}{n-2} - x\right| + \cdots = |x|.$$

Thus, we have:

$$B_n(Tx, Tx, \ldots, Tx, T_0) = |x|, \quad \max\{B_n(Tx, Tx, \ldots, Tx, x), B_n(T_0, T_0, \ldots, T_0, 0)\} = |x|.$$

This proves that T does not satisfy the condition of Corollary 2.10. However, we also have:

$$B_n(Tx, Tx, \ldots, Tx, Ty) \leqslant \frac{n-1}{n-2} B_n(x, x, \ldots, x, y) + \frac{n-3}{n-1} B_n(Tx, Tx, \ldots, Tx, x) + \frac{n-3}{n-1} B_n(Ty, Ty, \ldots, Ty, y).$$

Thus, T satisfies the condition of Corollary 2.12, and it is clear that T has a unique fixed-point at $x = 0$.

Corollary 2.6. *Let T be a self-map on a complete B_n-metric space (X, B_n), and suppose:*

$$B_n(Tx, Tx, \ldots, Tx, Ty) \leqslant h\left(B_n(Tx, Tx, \ldots, Tx, y) + B_n(Ty, Ty, \ldots, Ty, x)\right),$$

for some $h \in \left[0, \frac{n-3}{n-1}\right)$ and all $x, y \in X$. Then T has a unique fixed-point in X. Moreover, T is continuous at the fixed-point.

Proof. The assertion follows using Theorem 2.6 with:

$$M(x_1, x_2, \ldots, x_{n+1}) = h \max(x_{n-1}, x_n),$$

for some $h \in \left[0, \frac{n-3}{n-1}\right)$ and all $x_1, \ldots, x_{n+1} \in \mathbb{R}_+$. Indeed, M is continuous.

First, we have:

$$M(x_1, x_1, \ldots, 0, \ldots, x_{n-1}, x_{n-2}) = h \max(0, x_{n-1}).$$

If $x_{n-2} \leqslant M(x_1, x_1, \ldots, 0, \ldots, x_{n-1}, x_{n-2})$ with $x_{n-1} \leqslant 2x_{n-3} + x_{n-2}$, then:

$$x_{n-2} \leqslant \frac{2h}{1-h} x_{n-3}, \quad \text{where} \quad \frac{2h}{1-h} < 1.$$

Therefore, T satisfies condition (C1).

Next, if $x_{n-2} \leqslant M(x_{n-2}, 0, \ldots, x_{n-2}, x_{n-2}, \ldots, 0) = hy$, then $x_{n-2} = 0$, since $h < \frac{n-3}{n-1}$. Thus, T satisfies condition (C2).

Finally, if $x_i \leqslant y_i + z_i$ for $i \leqslant n+1$, then:

$$M(x_1, \ldots, x_{n+1}) = h\{x_{n-1}, x_n\} \leqslant h\{y_{n-1} + z_{n-1}, y_n + z_n\}$$

$$= h\{y_{n-1}, y_n\} + h\{z_{n-1}, z_n\} = M(y_1, \ldots, y_{n+1}) + M(z_1, \ldots, z_{n+1}).$$

Moreover:

$$M(0, 0, \ldots, 0, y, 2y) = h\{0, 2y\} = 2hy, \quad \text{where} \quad h < 1.$$

Therefore, T satisfies condition (C3). □

Corollary 2.7. *Let T be a self-map on a complete B_n-metric space (X, B_n), and suppose:*

$$B_n(Tx, Tx, \ldots, Tx, Ty) \leqslant a\left(B_n(Tx, Tx, \ldots, Tx, y) + B_n(Ty, Ty, \ldots, Ty, x)\right),$$

for some $a \in \left[0, \frac{n-3}{n-1}\right)$ and all $x_1, \ldots, x_{n+1} \in X$. Then T has a unique fixed-point in X. Moreover, T is continuous at the fixed-point.

PROOF. The assertion follows using Theorem 2.6 with:

$$M(x_1, x_2, \dots, x_{n+1}) = a(x_{n-1} + x_n),$$

for some $a \in \left[0, \frac{n-3}{n-1}\right)$ and all $x_1, \dots, x_{n+1} \in \mathbb{R}_+$. Indeed, M is continuous.

First, we have:

$$M(x_1, x_1, \dots, 0, \dots, x_{n-1}, x_{n-2}) = a(0 + x_{n-1}) = ax_{n-1}.$$

If $x_{n-2} \leqslant M(x_1, x_1, \dots, 0, \dots, x_{n-1}, x_{n-2})$ with $x_{n-1} \leqslant 2x_{n-3} + x_{n-2}$, then:

$$x_{n-2} \leqslant \frac{2a}{1-a} x_{n-3}, \quad \text{where} \quad \frac{2a}{1-a} < 1.$$

Therefore, T satisfies condition (C1).

Next, if $y \leqslant M(x_{n-2}, 0, \dots, x_{n-2}, x_{n-2}, \dots, 0) = a(x_{n-2} + x_{n-2}) = 2ax_{n-2}$, then $x_{n-2} = 0$ since $2a < \frac{n-2}{n-1}$. Thus, T satisfies condition (C2).

Finally, if $x_i \leqslant y_i + z_i$ for $i \leqslant n+1$, then:

$$M(x_1, \dots, x_{n+1}) = a(x_{n-1} + x_n) \leqslant a\{y_{n-1} + z_{n-1}, y_n + z_n\}$$

$$= a\{y_{n-1}, y_n\} + a\{z_{n-1}, z_n\} = M(y_1, \dots, y_{n+1}) + M(z_1, \dots, z_{n+1}).$$

Moreover:

$$M(0, 0, \dots, 0, x_{n-2}, 2x_{n-2}) = a(0 + x_{n-2}) = ax_{n-2}, \quad \text{where} \quad a < 1.$$

Therefore, T satisfies condition (C3). □

EXAMPLE 2.4. Let $\mathbb{R}$ be the usual B_n-metric space as in Example 1.7, and define:

$$T(x) = \frac{x}{n-1}, \quad \text{for all } x \in [0, 1].$$

Then we have:

$$B_n(Tx, Tx, \dots, Tx, Ty) = (n-2)\left|\frac{x}{n-1} - \frac{y}{n-1}\right| = \frac{n-2}{n-1}|x - y|,$$

$$B_n(Tx, Tx, \dots, Tx, y) = (n-2)\left|\frac{x}{n-1} - y\right|,$$

$$B_n(Ty, Ty, \dots, Ty, x) = (n-2)\left|\frac{y}{n-1} - x\right|.$$

It implies that:

$$B_n(T_1, T_1, \dots, T_1, T_0) = \frac{n-2}{n-1} B_n(T_1, T_1, \dots, T_1, 0) = \frac{n-2}{n-1} B_n(T_0, T_0, \dots, T_0, 1) = n-2.$$

This proves that T does not satisfy the condition of Corollary 2.14.

We also have:

$$B_n(Tx, Tx, \dots, Tx, y) + B_n(Ty, Ty, \dots, Ty, x) = (n-2)\left|\frac{x}{n-1} - y\right| + (n-2)\left|\frac{y}{n-1} - x\right| \geqslant \frac{8}{n-1}|x - y|.$$

Therefore, T satisfies the condition of Corollary 2.15, and it is clear that T has a unique fixed-point at $x = 0$.

COROLLARY 2.8. *Let T be a self-map on a complete B_n-metric space (X, B_n), and suppose:*

$$B_n(Tx, Tx, \dots, Tx, Ty) \leqslant aB_n(x, x, \dots, x, y) + bB_n(Tx, Tx, \dots, Tx, y) + cB_n(Ty, Ty, \dots, Ty, x),$$

for some $a, b, c \geqslant 0$, $a + b + c < 1$, and $a + 3c < 1$, and for all $x, y \in X$. Then T has a unique fixed-point in X. Moreover, T is continuous at the fixed-point.

PROOF. The assertion follows using Theorem 2.6 with:

$$M(x_1, x_2, \ldots, x_{n+1}) = ax_1 + bx_{n-1} + cx_n,$$

for some $a, b, c \geqslant 0$, with $a + b + c < 1$ and $a + 3c < 1$ and all $x_1, \ldots, x_{n+1} \in \mathbb{R}_+$. Indeed, M is continuous.

First, we have:

$$M(x_1, x_1, \ldots, 0, \ldots, x_{n-1}, x_{n-2}) = ax_1 + cx_{n-1}.$$

If $x_{n-2} \leqslant M(x_1, x_1, \ldots, 0, \ldots, x_{n-1}, x_{n-2})$ with $x_{n-1} \leqslant 2x_{n-3} + x_{n-2}$, then:

$$x_{n-2} \leqslant \frac{a+2c}{1-c} x_{n-3}, \quad \text{where} \quad \frac{a+2c}{1-c} < 1.$$

Therefore, T satisfies condition (C1).

Next, if $x_{n-2} \leqslant M(x_{n-2}, 0, \ldots, x_{n-2}, x_{n-2}, \ldots) = (a + b + c)x_{n-2}$, then $x_{n-2} = 0$, since $a + b + c < 1$. Therefore, T satisfies condition (C2).

Finally, if $x_i \leqslant y_i + z_i$ for $i \leqslant n + 1$, then:

$$M(x_1, \ldots, x_{n+1}) = ax_{n-3} + bx_{n-1} + cx_n \leqslant a(y_{n-3} + z_{n-3}) + b(y_{n-1} + z_{n-1}) + c(y_n + z_n)$$

$$= (ay_{n-3} + by_{n-1} + cy_n) + (az_{n-3} + bz_{n-1} + cz_n) = M(y_1, \ldots, y_{n+1}) + M(z_1, \ldots, z_{n+1}).$$

Moreover:

$$M(0, 0, \ldots, 0, y, 2y) = cy, \quad \text{where} \quad c < 1.$$

Therefore, T satisfies condition (C3). □

EXAMPLE 2.5. Let $\mathbb{R}$ be the usual B_n-metric space as in Example 1.7, and define:

$$T(x) = \frac{n-1}{n(1-x)}, \quad \text{for all } x \in [0, 1].$$

Then we have:

$$B_n(Tx, Tx, \ldots, Tx, Ty) = \frac{n-1}{n-2}|x - y|,$$

$$B_n(Tx, Tx, \ldots, Tx, y) = \frac{n-2}{n}|n - 1/n(1 - x) - y|.$$

It implies that:

$$B_n(T_1, T_1, \ldots, T_1, T_0) = \frac{n-1}{n-2}, \quad \max\{B_n(T_1, T_1, \ldots, T_1, 0), B_n(T_0, T_0, \ldots, T_0, n-3)\} = \frac{n-3}{n-2}.$$

This proves that T does not satisfy the condition of Corollary 2.14. We also have:

$$\frac{n}{n+1} B_n(Tx, Tx, \ldots, y) + 0 \cdot B_n(Tx, Tx, \ldots, Ty) + 0 \cdot B_n(Ty, Ty, \ldots, x) = \frac{n}{n+1}|x - y| \geqslant B_n(Tx, Tx, \ldots, Ty).$$

Therefore, T satisfies the condition of Corollary 2.17. It is clear that T has a unique fixed-point at:

$$x = \frac{n-1}{7}.$$

COROLLARY 2.9. *Let T be a self-map on a complete B_n-metric space (X, B_n), and suppose:*

$$B_n(Tx, Tx, \ldots, Tx, Ty) \leqslant a_1 B_n(x, x, \ldots, x, y) + a_2 B_n(Tx, Tx, \ldots, Tx, x) + \cdots + a_{n+1} B_n(Ty, Ty, \ldots, Ty, y),$$

for some $a_1, \ldots, a_{n+1} \geqslant 0$ such that:

$$\max\{a_1 + a_2 + \cdots + a_{n-1} + a_n + a_{n+1}, a_1 + \cdots + a_{n-1} + a_n, a_n + 2a_{n+1}\} < 1,$$

and for all $x, y \in X$. Then T has a unique fixed-point in X. Moreover, T is continuous at the fixed-point.

PROOF. The assertion follows using Theorem 2.6 with:

$$M(x_1, \ldots, x_{n+1}) = a_1x_1 + a_2x_2 + \cdots + a_nx_n + a_{n+1}x_{n+1},$$

for some $a_1, \ldots, a_{n+1} \geqslant 0$ such that:

$$\max\{a_1 + a_2 + \cdots + a_{n-1} + a_n + a_{n+1}, a_1 + \cdots + a_{n-1} + a_n, a_n + 2a_{n+1}\} < 1.$$

First, we have:

$$M(x_1, x_1, \ldots, 0, \ldots, x_{n-1}) = a_1x_1 + a_2x_2 + \cdots + a_nx_{n-1} + a_{n+1}x_2.$$

If $x_{n-2} \leqslant M(x_1, x_1, \ldots, 0, \ldots, x_{n-1})$ with $x_{n-2}x_{n-1} \leqslant 2x_{n-3} + x_{n-2}$, then:

$$x_{n-2} \leqslant \frac{a_1 + a_2 + \cdots + a_{n-2}}{1 - a_n - a_{n+1}} x_{n-3}, \quad \text{where} \quad \frac{a_1 + a_2 + \cdots + a_{n-2}}{1 - a_n - a_{n+1}} < 1.$$

Therefore, T satisfies condition (C1).

Next, if $x_{n-2} \leqslant M(x_{n-2}, 0, \ldots, x_{n-2})$ with $x_{n-2} = 0$, then T satisfies condition (C2). Similarly, for condition (C3), we show that T satisfies the necessary inequalities. □

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