On pseudo slant submanifolds of (k, μ) contact manifold

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Abstract:

This work "On Pseudo Slant Submanifolds of (k, μ) Contact Manifold" has endeavored to establish the requirements for integrating the distributions of pseudo slant submanifolds within a (k, μ) contact manifold. The aim of this work is to study on the geometry of pseudo-slant submanifolds having the background of kenmotsu manifolds; generalizing sasakian space forms. Also to suggest fresh insights into their integrability requirements and associated features.

In the study, an attempt is served to identify the integrability criteria for (k, μ) contact manifolds, which will enhance an understanding about their geometric structures and applications. The findings from this study has given few significant contributions around differential geometry, in particular studying on geometric structures within (k, μ) contact manifolds. Also, create significant contributions to varied areas of mathematics, physics, and engineering. Some of the practical applications from these findings may be applied for the potential use in theoretical physics, in the study of space time geometry and which would create the way for further research in similar domains such as gravitational theory and cosmology. The established scope of integrability criteria will also result in fresh uses in computational geometry, while offering crucial information for the advancement of modeling and algorithmic approaches.

The detailed investigation of integrability requirements in this research article advances knowledge of geometric structures in (k, μ) contact manifolds. Further, it places the groundwork *for further developments and applications, preparing the scientific community for additional study in related fields.*

Key words - Pseudo slant submanifolds, manifold, contact manifold, (k, μ) contact manifolds, slant submanifolds, pseudo slant submanifolds, integrability conditions.

1 Preface

In 1990, Chen [2] presented a slant immersion system. Numerous analysts, including Lotta [1] on slant immersions and Cabrerizo et al [4, 3] on K-contact and Sasakian manifolds, have produced useful results in this area. A. Carrizo [6] proposed hemislant submanifolds, which were later renamed pseudoslant submanifolds. Suleyman Dirik and Mehmet Atceken examined the geometry of pseudo-slant submanifolds around a Kenmotsu manifold [7]. Avijit Saskar and U. C. De investigated the integrability requirements of T.S.M. distributions under P.S.S.M. [8]. Bhavya K and Somashekhara G [12] have a useful result for pseudo-slant submanifolds in generalised sasakian space form with integrability constraints [5]. This work provides the integrability conditions on (k,μ) contact manifolds.

The study's objective was to:

- Investigate and develop integrability criteria for pseudo slant submanifolds in a (k, μ) contact manifold.
- Examine the geometry of pseudo-slant submanifolds in a Kenmotsu manifold.
- Generalize Sasakian space forms and provide integrability criteria for related pseudo slant submanifolds.
- Investigate the integrability of distributions for pseudo slant submanifolds on (k, μ) contact manifolds.

Methodology of the study:

- The (k, μ) contact manifold's virtually contact metric structure was meticulously examined, including the vector field, tensor field, and 1-form, to ensure adherence to Riemannian metric constraints.
- Investigating the integrability of distributions for pseudo slant submanifolds in (k, μ) contact manifolds.
- Using mathematical formulas like Gauss and Weingarten can help to establish the integrability of pseudo slant submanifolds in specific geometric contexts.

2 Preliminaries

Let us consider $(\overline{M}, g)$ a manifold accomplished with dimension $(2n+1)$ furnished of almost contact metric structure (ϕ, ξ, η, g) which has a vector field (v.f.) ξ, (1,1) tensor field ϕ, 1-form η as well as g being a Riemannian metric fulfilling the following,

$$\phi^2 X_3 = -X_3 + \eta(X_3)\xi, \eta(X_3) = g(X_3, \xi), \eta(\xi) = 1, \tag{1}$$

$$g(\phi X_3, \phi Y_3) = g(X_3, Y_3) - \eta(X_3)\eta(Y_3), \tag{2}$$

$$g(\phi X_3, Y_3) + g(X_3, \phi Y_3) = 0, \phi\xi = 0, \eta(\phi X_3) = 0, \tag{3}$$

$\forall$ v.f. $X_3, Y_3 \in T\overline{M}$.

We conclude that almost contact metric manifold(a.c.m.m.) $\overline{M}$ is termed as sasakian manifold whenever the following equation is satisfied.

$$(\overline{\nabla}_{X_1}\phi)Y_3 = g(X_3, Y_3)\xi - \eta(Y_3)X_3, \tag{4}$$

$$\overline{\nabla}_{X_1}\xi = -\phi X_3 \tag{5}$$

where $\overline{\nabla}$ indicates the Levi-Civita connection with regard to g.

Consider M, as a n-dim. Riemannian manifold(R.m.) amidst metric g isometrically immersed in $\overline{M}$, then ∇ and $\overline{\nabla}$ indicates the Levi-civita connection of M as well as $\overline{M}$ respectively. TM as well as $T^{\perp}M$ are the tangent bundle and the normal bundle on M. Express the collection of all V.f. perpendicular to M by $T^{\perp}M$.

$\forall X_3 \in TM$ as well as $N \in T^{\perp}M$, we draft as

$$\phi X_3 = TX_3 + NX_3, \tag{6}$$

where NX_3 and TX_3 are the normal and tangential components of ϕX_3 respectively.

$$\phi N = tN + nN. \tag{7}$$

where nN as well as tN represents the normal as well as tangential factors with respect to ϕN.

The Gauss and Weingarten formulae are likely to be,

$$\bar{\nabla}_{X_3} Y_3 = \nabla_{X_3} Y_3 + h(X_3, Y_3), \tag{8}$$

$$\bar{\nabla}_{X_3} N = \nabla^{\perp}_{X_3} N - A_N X_3. \tag{9}$$

for any $X_3, Y_3 \in TM$. The conjunction in the Normal bundle is represented by $\nabla^{\perp}$. A_N being Weingarten endomorphism correlated with N along with h being the second fundamental form belongs to M associated with the shape operator A by following,

$$g(h(X_3, Y_3), N) = g(A_N X_3, Y_3). \tag{10}$$

From (8), we have

$$\bar{\nabla}_{X_3} \xi = \nabla_{X_3} \xi + h(X_3, \xi). \tag{11}$$

An a.c.m. will be Trans-Sasakian manifold(T.S.M.) if

$$(\bar{\nabla}_{X_3} \phi) Y_3 = \alpha[g(X_3, Y_3)\xi - \eta(Y_3) X_3] + \beta[g(\phi X_3, Y_3)\xi - \eta(Y_3)\phi X_3], \tag{12}$$

An a.c.m. is termed as (k,μ) contact metric manifold in case that

$$(\bar{\nabla}_{X_3} \phi) Y_3 = g(X_3 + \sigma X_3, Y_3)\xi - \eta(Y_3)(X_3 + \sigma X_3), \tag{13}$$

and

$$(\bar{\nabla}_{X_3} \xi) =- \phi X_3 - \phi\sigma X_3, \tag{14}$$

Consequently, (8), (9) along with (12), (6) and (7) signifies that

$$(\nabla_{X_3} T) Y_3 = A_{NY_3} X_3 + th(X_3, Y_3) + g(Y_3, X_3 + \sigma X_3)\xi - \eta(Y_3)(X_3 + \sigma X_3), \tag{15}$$

$$(\nabla_{X_3}N)Y_3 = nh(X_3, Y_3) - h(X_3, TY_3), \tag{16}$$

Using (14), equating tangential and normal components we acquire,

$$\nabla_{X_3}\xi = -\phi X_3 - \phi\sigma X_3, \tag{17}$$

$$h(X_3, \xi) = 0. \tag{18}$$

The mean curvature vector which stands for H with regard to M can be expressed as

$$H = \frac{1}{n}trace(h) = \frac{1}{n}\sum h(e_i, e_i). \tag{19}$$

where n stands for the dimension of M and $e_i = \{e_1, e_2, \ldots\ldots, e_{n\}}$ indicates a local orthonormal frame of M.

If the submanifold M is totally umbilical then,

$$h(X_3, Y_3) = g(X_3, Y_3)H. \tag{20}$$

Further the submanifold M is termed as totally geodesic if it fulfills the condition that $h(X_3, Y_3) = 0, \forall X_3, Y_3 \in \Gamma(TM)$ as well as M is called minimal if $h = 0$.

The covariant derivatives of T and Y_3 are determined as

$$(\nabla_{X_3}T)Y_1 = \nabla_{X_3}TY_3 - T\nabla_{X_3}Y_3, \quad \forall X_3, Y_3 \in \Gamma(TM), \tag{21}$$

$$(\nabla_{X_3}N)Y_3 = \nabla^{\perp}_{X_3}NY_3 - N\nabla_{X_3}Y_3, \quad \forall X_3, Y_3 \in \Gamma(TM). \tag{22}$$

We also obtain the following equalities by direct calculations.

$$(\nabla_{X_3}T)Y_3 = A_{NY_3}X_3 + th(X_3, Y_3) + g(Y_3, X_3 + \sigma X_3)\xi - \eta(Y_3)(X_3 + \sigma X_3), \tag{23}$$

and

$$(\nabla_{X_3}N)Y_3 = nh(X_3, Y_3) - h(X_3, TY_3). \tag{24}$$

Put $Y_1 = \xi$ in (13) and using (1), (2) and (3) we get,

$$(\overline{\nabla}_{X_3}\phi)\xi = g(X_3 + \sigma X_3, \xi)\xi - (X_3 + \sigma X_3), \tag{25}$$

$$= \eta(X_3 + \sigma X_3)\xi - (X_3 + \sigma X_3). \tag{26}$$

Using covariant derivative in LHS and using (1), (2) and (3) we get,

$$(\overline{\nabla}_{X_3}\phi)\xi = -\phi\overline{\nabla}_{X_3}\xi.$$

Therefore (25) implies,

$$T\nabla_{X_3}\xi + N\nabla_{X_3}\xi = (X_3 + \sigma X_3) - \eta(X_3 + \sigma X_3)\xi. \tag{27}$$

Equalizing the tangential as well as normal components with respect to (27) we acquire,

$$T\nabla_{X_3}\xi = X_3 - \eta(X_3)\xi, \tag{28}$$

$$N\nabla_{X_3}\xi = \sigma X_3 - \eta(\sigma X_3)\xi. \tag{29}$$

Definition 2.1:

"Consider M as a submanifold pertaining to (k,μ) contact manifold $\overline{M}$. The angle $\theta(l) \in [0,\frac{\pi}{2}]$ is the slant angle concerning to M composed for every non-zero vector X_3 tangent to M at u.

1. The submanifold[s.m.] can be labelled as slant submanifold(S.S.M.) if θ is constant, for $X_3 \in \Gamma(TM)$.
2. Suppose $\theta = 0$, the s.m. is called invariant submanifold(I.S.M.).
3. Suppose $\theta = \frac{\pi}{2}$, the s.m. is anti-invariant submanifold.
4. Suppose $\theta(l) \in (0,\frac{\pi}{2})$,the s.m. is proper slant submanifold [9]".

Consider M as a slant submanifold(S.S.M.) of an a.c.m. $\overline{M}$,implies that the T.b. TM of M can be resolved as

$$TM = D_\theta \oplus < \xi >, \tag{30}$$

where $< \xi >$ indicates the distribution connected by the structure V.f. ξ also the complementary of distribution of $< \xi >$ is represented byD_θ in TM which is termed as the slant distribution with regard to M.

Considering a proper S.S.M. M with regard to a.c.m. $\overline{M}$ along with the slant angle θ, Lotta [10], established that

$$T^2X_3 =- cos^2\theta[X_3 - \eta(X_3)\xi].$$

$\forall X_3 \in \Gamma(TM)$.

From [11] given by Cabrerizo et al. in which characterisation for a S.S.M. in a c.m.m. is investigated we acquire the subsequent axioms

If M is a S.S.M. of an a.c.m. $\overline{M}$ provided $\xi \in \Gamma(TM)$, then M is S.S.M. iff. there exists a consistent $\lambda \in [0,1]$ such that

$$T^2(X_3) =- \lambda(X_3 - \eta(X_3)\xi). \tag{31}$$

Additionally, the slant angle θ of M is related as follows $\lambda = cos^2\theta$ [11].

Corollary 2.3: If M is a S.S.M. of an a.c.m. $\overline{M}$ with slant angle θ, then for any $X_3, Y_3 \in \Gamma(TM)$, we admit

$$g(TX_3, TY_3) = cos^2\theta[g(X_3, Y_3) - \eta(X_3)\eta(Y_3)],$$

$$g(NX_3, NX_3) = sin^2\theta[g(X_3, Y_3) - \eta(X_3)\eta(Y_3)].$$

3 Pseudo-slant submanifolds [P.S.S.M.] of a (k, μ) contact manifold

In this part, the integrability conditions of the distributions of P.S.S.M. of a (k, μ) contact manifold is accessed in continuation of [12].

Consider two projections P_1 and P_2 on $D^\perp$ and D_θ respectively, then for any V.f. $X_3 \in \Gamma(TM)$ we have,

The definition 2.1 holds good even for (k, μ) contact manifold.

$$X_3 = P_1X_3 + P_2X_3 + \eta(X_3)\xi. \tag{32}$$

Now employing ϕ on both sides of (32), we get

$$\phi X_3 = \phi P_1X_3 + \phi P_2X_3, \tag{33}$$

$$TX_3 + NX_3 = NP_1X_3 + TP_2X_3 + NP_2X_3. \tag{34}$$

Equating tangential and normal components

$$TX_3 = TP_2X_3, \;\; NX_3 = NP_1X_3 + NP_2X_3. \tag{35}$$

and

$$\phi P_1X_3 = NP_1X_3, \; TP_1X_3 = 0, \phi P_2X_3 = TP_2X_3 + NP_2X_3, TP_2X_3 \in \Gamma(D_\theta). \tag{36}$$

Let us indicate the orthogonal complementary of ϕTM in $T^{\perp}M$ by μ, then the N.b. $T^{\perp}M$ is resolved as follows

$$T^{\perp}M = N(D^{\perp}) \oplus N(D_{\theta}) \oplus \mu, \tag{37}$$

μ represents an invariant sub-bundle of $T^{\perp}M$ as $N(D^{\perp})$ also $N(D_{\theta})$ represents the orthogonal distribution with respect to $\overline{M}$, $g(X_3, L_6) = 0, \forall L_6 \in \Gamma(D^{\perp})$ as well as $X_3 \in \Gamma(D_{\theta})$.

Thus by (??), (??) and (6)

$$g(NL_6, NX_3) = g(\phi L_6, \phi X_3) = g(L_6, X_3) = 0. \tag{38}$$

The distributions $N(D_{\theta})$ as well as $N(D^{\perp})$ are mutually perpendicular. Further (37) is an orthogonal direct decomposition.

Let M be a submanifold of an a.c.m. $\overline{M}$, implies that D_{θ} is slant distribution about M iff. there is a constant $\lambda \in [0,1]$ provided

$$(TP_2)^2 X_3 =- \lambda X_3, \tag{39}$$

$\forall X_3 \in \Gamma(D_{\theta})$, in such case, θ being the slant angle with regard to M, then $\lambda = Cos^2\theta$ [11].

If M is a P.S.S.M. of a (k, μ) contact manifold $\overline{M}$, implies that the anti-invariant distribution $D^{\perp}$ is integrable iff.

$$A_{NX_3}Y_3 = A_{NY_3}X_3, \tag{40}$$

for any $X_3, Y_3 \in \Gamma(D^{\perp})$ such that

$$g(\sigma Y_3, X_3)\xi - g(\sigma X_3, Y_3)\xi = 0. \tag{41}$$

Proof: By using (13), (8), (9), (20) and (6) we have,

$$(\overline{\nabla}_{X_3}\phi)Y_3 = \overline{\nabla}_{X_3}\phi Y_3 - \phi\overline{\nabla}_{X_3}Y_3,$$

$$g(X_3 + \sigma X_3, Y_3)\xi - \eta(Y_3)(X_3 + \sigma X_3) = \overline{\nabla}_{X_3}NY_3 - \phi[\nabla_{X_3}Y_3 + h(X_3, Y_3)],$$

$$\begin{aligned} g(X_3, Y_3)\xi + g(\sigma X_3, Y_3)\xi \quad &=- A_{NY_3}X_3 + \nabla^{\perp}_{X_3}NY_3 - T(\nabla_{X_3}Y_3) - N(\nabla_{X_3}Y_3) \\ &-th(X_3, Y_3) - nh(X_3, Y_3), \end{aligned}$$

for any $X_3, Y_3 \in \Gamma(D^{\perp})$.

Equalizing tangent and normal components in the previous equation we obtain,

$$- A_{NY_3}X_3 - T(\nabla_{X_3}Y_3) - th(X_3, Y_3) = g(X_3, Y_3)\xi + g(\sigma X_3, Y_3)\xi, \tag{42}$$

$$\nabla^{\perp}_{X_3} NY_3 - N(\nabla_{X_3} Y_3) - nh(X_3, Y_3) = 0. \tag{43}$$

Interchange X_3 and Y_3 in (42) we get

$$-A_{NX_3} Y_3 - T(\nabla_{Y_3} X_3) - th(X_3, Y_3) = g(X_3, Y_3)\xi + g(\sigma Y_3, X_3)\xi. \tag{44}$$

Now from (44) and (42) we get,

$$A_{NY_3} X_3 - A_{NX_3} Y_3 + T[\nabla_{X_3} Y_3 - \nabla_{Y_3} X_3] = g(\sigma Y_3, X_3)\xi - g(\sigma X_3, Y_3)\xi,$$

$$T[X_3, Y_3] = g(\sigma Y_3, X_3)\xi - g(\sigma X_3, Y_3)\xi + A_{NX_3} Y_3 - A_{NY_3} X_3. \tag{45}$$

From (41)

$$T[X_3, Y_3] = A_{NX_3} Y_3 - A_{NY_3} X_3. \tag{46}$$

Conversely: If (46) holds good then, from (45) we obtain, (41).

Let M be a P.S.S.M. of a (k, μ) contact manifold $\overline{M}$, then the anti-invariant distribution $D^{\perp}$ is integrable iff.

$$(\nabla_{L_3} T)L_6 = (\nabla_{L_6} T)L_3, \tag{47}$$

for any $L_3, L_6 \in \Gamma(D^{\perp})$.

Proof: In consideration of the ambient manifold $\overline{M}$ which is a (k, μ) contact manifold, for any $L_3, L_6 \in \Gamma(D^{\perp})$ we get,

$$(\overline{\nabla}_{L_3} \phi)L_6 = g(L_3 + \sigma L_3, L_6)\xi - \eta(L_6)(L_3 + \sigma L_3). \tag{48}$$

Equivalent to,

$$\overline{\nabla}_{L_3} \phi L_6 - \phi \overline{\nabla}_{L_3} L_6 = g(L_3 + \sigma L_3, L_6)\xi - \eta(L_6)(L_3 + \sigma L_3). \tag{49}$$

From (8)

$$\overline{\nabla}_{L_3} NL_6 - \phi[\nabla_{L_3} L_6 + h(L_6, L_3)] = g(L_3, L_6)\xi + g(\sigma L_3, L_6)\xi - \eta(L_6)(L_3 + \sigma L_3). \tag{50}$$

Using (9) and (6) in the LHS of (50)

$$-A_{NL_6} L_3 + \nabla^{\perp}_{L_3} NL_6 - T\nabla_{L_3} L_6 - th(L_6, L_3) - N\nabla_{L_3} L_6 - nh(L_6, L_3) = g(L_3, L_6)\xi + g(\sigma L_3, L_6)\xi - \eta(L_6)(L_3 + \sigma L_3). \tag{51}$$

Since $\xi \in \Gamma TM$, $L_6 \in \Gamma D^{\perp}$, $\eta(L_6) = 0$

Relating tangential and normal factors we acquire,

$$-[A_{NL_6} L_3 + T\nabla_{L_3} L_6 + th(L_6, L_3)] = g(L_3, L_6)\xi + g(\sigma L_3, L_6)\xi, \tag{52}$$

$$\nabla^{\perp}_{L_3} NL_6 - N\nabla_{L_3} L_6 - nh(L_6, L_3) = 0. \tag{53}$$

From (42), (44) and Theorem (7.6.2)

$$T[L_6, L_3] = A_{NL_6} L_3 + T(\nabla_{L_3} L_6) + th(L_6, L_3), \tag{54}$$

for $[L_3, L_6] \in \Gamma D^{\perp}$, $\phi[L_3, L_6] = N[L_3, L_6]$ because of the tangent factor of $\phi[L_3, L_6] = 0$. So we get

$$A_{NL_6} L_3 + T(\nabla_{L_6} L_3) + th(L_6, L_3) = 0. \tag{55}$$

Similarly we obtain

$$A_{NL_3} L_6 + T(\nabla_{L_3} L_6) + th(L_6, L_3) = 0. \tag{56}$$

By Theorem (7.6.2), (55) and (56)

$$T(\nabla_{L_6} L_3) = T(\nabla_{L_3} L_6). \tag{57}$$

Thus the anti-invariant distribution $D^{\perp}$ is integrable iff. (47) is satisfied.

Let M be P.S.S.M. of a (k, μ) contact manifold $\overline{M}$, then

$$\eta([X_3, Y_3]) = 0, \tag{58}$$

provided

$$g(\phi X_3 + \phi\sigma X_3, Y_3) = 0, \tag{59}$$

for any $X_3, Y_3 \in \Gamma(D^{\perp} \oplus D_{\theta})$.

Proof: In consideration of the ambient space which is a (k, μ) contact manifold for any $X_3, Y_3 \in \Gamma(D^{\perp} \oplus D_{\theta})$ we retain

$$g([X_3, Y_3], \xi) = g(\overline{\nabla}_{Y_3} \xi, X_3) - g(\overline{\nabla}_{X_3} \xi, Y_3). \tag{60}$$

Using (8)

$$g([X_3, Y_3], \xi) = g(\nabla_{Y_3} \xi, X_3) - g(\nabla_{X_3} \xi, Y_3). \tag{61}$$

Using (17)

$$g([X_3, Y_3], \xi) = g(-\phi Y_3 - \phi\sigma Y_3, X_3) - g(-\phi X_3 - \phi\sigma X_3, Y_3), \tag{62}$$

$$g([X_3, Y_3], \xi) = 2g(\phi X_3 + \phi\sigma X_3, Y_3). \tag{63}$$

Inview of (59) we obtain

$$\eta([X_3, Y_3]) = 0.$$

Let M be a P.S.S.M. of a (k, μ) contact manifold $\overline{M}$, then the slant distribution D_{θ} is integrable if

$$\eta(Y_3)P_1(X_3 + \sigma X_3) = P[T\nabla_{Y_3}X_3 - \nabla_{X_3}TY_3 + A_{NY_3}X_3 + th(X_3, Y_3)], \quad (64)$$

for any $X_3, Y_3 \in \Gamma(D_\theta)$.

Proof: By using (23), considering the tangential component

$$(\nabla_{X_3}T)Y_3 = A_{NY_3}X_3 + th(X_3, Y_3) + g(Y_3, X_3 + \sigma X_3)\xi - \eta(Y_3)(X_3 + \sigma X_3).$$

Also,

$$T\nabla_{X_3}Y_3 = T[X_3, Y_3] + T\nabla_{Y_3}X_3. \quad (65)$$

Using (21) and (65), we get

$$(\nabla_{X_3}T)Y_3 = \nabla_{X_3}TY_3 - T[X_3, Y_3] - T\nabla_{Y_3}X_3. \quad (66)$$

From (23) and (66) we have,

$$T[X_3, Y_3] = \nabla_{X_3}TY_3 - T\nabla_{Y_3}X_3 - A_{NY_3}X_3 - th(X_3, Y_3) - g(Y_3, X_3 + \sigma X_3)\xi + \eta(Y_3)(X_3 + \sigma X_3). \quad (67)$$

Applying P_1 to (67) for any $X_3, Y_3 \in \Gamma(D_\theta)$, we get (64).

Consider M as a P.S.S.M. of a (k, μ) contact manifold $\overline{M}$ implies that the distribution $D^\perp \oplus < \xi >$ is constantly integrable.

Proof: We admit by sincere computation

$$g([X_3, \xi], TL_6) = g(\overline{\nabla}_{X_3}\xi, TL_6) - g(\overline{\nabla}_\xi X_3, TL_6), \quad (68)$$

for every $X_3 \in \Gamma(D^\perp)$ also $L_6 \in \Gamma(D_\theta)$ where $D^\perp$ as well as D_θ are the two orthogonal distributions and $D^\perp$ being an anti-invariant. Inview of (11), (17), (18) and (62), we obtain

$$g([X_3, \xi], TL_6) = - g(\overline{\nabla}_\xi X_3, TL_6) = g(\nabla_\xi TL_6, X_3). \quad (69)$$

Since $X_3 \in \Gamma D^\perp$, $L_6 \in \Gamma D_\theta$ implies,

$$g([X_3, \xi], TL_6) = 0. \quad (70)$$

So $[X_3, \xi] \in \Gamma(D^\perp)$ for $X_3 \in \Gamma(D^\perp)$.

Hence the distribution $D^\perp \oplus < \xi >$ is constantly integrable.

Consider M as a proper P.S.S.M. with regard to a (k, μ) contact manifold $\overline{M}$. If N is parallel about D_θ, implies that either M is a D_θ geodesic submanifold or $h(X_3, Y_3)$ being an eigen vector with respect to n^2 along with eigen value $-cos^2\theta$.

Proof: Since $(\nabla_{X_3}N)Y_3 = 0$, for any $X_3, Y_3 \in \Gamma(D_\theta)$ from (24), we admit

$$nh(X_3, Y_3) - h(X_3, TY_3) = 0. \quad (71)$$

Considering that D_θ being slant distribution $T\xi = 0$ along with $h(X_3, \xi) = 0$ we acquire

$$nh(X_3, Y_3 - \eta(Y_3)\xi) - h(X_3, T(Y_3 - \eta(Y_3)\xi) = 0, \tag{72}$$

$$nh(X_3, Y_3 - \eta(Y_3)\xi) = h(X_3, TY_3). \tag{73}$$

Employing n to (73) we retain,

$$n^2h(X_3, Y_3 - \eta(Y_3)\xi) = nh(X_3, TY_3). \tag{74}$$

On the otherhand, by altering Y_3 to TY_3 in (71) we obtain

$$nh(X_3, TY_3) = h(X_3, T^2Y_3). \tag{75}$$

Since M is a S.S.M., we have the following result

$$T^2Y_3 =- Cos^2\theta[Y_3 - \eta(Y_3)]\xi. \tag{76}$$

Applying (31) in (75)

$$nh(X_3, TY_3) =- Cos^2\theta h(X_3, Y_3 - \eta(Y_3)\xi). \tag{77}$$

Hence we get

$$n^2h(X_3, Y_3 - \eta(Y_3)\xi) = nh(X_3, TY_3) = h(X_3, T^2Y_3) =- Cos^2\theta h(X_3, Y_3 - \eta(Y_3)\xi). \tag{78}$$

This signifies that h either vanishes with respect to D_θ or h being an eigen vector with respect to n^2 along with the eigen value $-cos^2\theta$.

FUNDING: The authors disclose that no funding, grants, or other assistance was offered during the creation of this publication.

ETHICAL STATEMENT: This content is the author's original work and has not been previously published. The manuscript is not presently under consideration for publication elsewhere.

CONFLICTS OF INTEREST: The authors have no conflicts of interest in this paper.

DATA AVAILABILITY STATEMENT: The data set information is supplied in the publication that supports these study findings.

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