CERTAIN INVESTIGATION ON GENERALIZED SASAKIAN SPACE FORM AND COMPLEX SPACE FORM

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Abstract. In this paper, we explore curvature properties of generalized Sasakian space form, such as being conformally flat generalized Sasakian space form is generalized Sasakian space form provided with some conditions and locally symmetric generalized complex space form has constant scalar curvature.

1. introduction

The concept of a generalized Sasakian space form was introduced and examined by P. Alegre, D. Blair, and A. Carriazo [1], including several examples. A generalized Sasakian space form is defined as an almost contact metric manifold (M, ϕ, ξ, η, g) with a curvature tensor given by

$$\begin{aligned} R(X,Y)Z =& f_1(g(Y,Z)X - g(X,Z)Y) \\ &+ f_2(g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z) \\ &+ f_3(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi), \end{aligned} \tag{1.1}$$

where f_1,f_2 and f_3 differentiable functions on M and X,Y, Z are vector fields on M.
If (M, J, g) is a kaehlerian manifold with constant holomorphic sectional curvatures $K(X \bigwedge JX) = c$, then it is said to be complex space form and its curvature tensor is given by

$$\overline{R}(X,Y)Z = \frac{c}{4}(g(Y,Z)X - g(X,Z)Y + g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ). \tag{1.2}$$

More generally if the curvature tensor of an almost Hermitian manifold M satisfies

$$\overline{R}(X,Y)Z = F_1[g(Y,Z)X - g(X,Z)Y] + F_2[g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ]. \tag{1.3}$$

F_1 and F_2 being differentiable functions then M is said to be generalised complex space form.

$$\overline{S}(Y,Z) = \frac{c}{4}(n+2)g(Y,Z), \tag{1.4}$$

$$\overline{Q}Y = \frac{c}{4}(n+2)Y, \tag{1.5}$$

$$\overline{r} = \frac{c}{4}n(n+2), \tag{1.6}$$

An odd dimensional Riemannian manifold (M, g) is defined as an almost contact metric manifold if there exists a $(1,1)$ tensor field ϕ, a vector field ξ (known as the structure vector field), and a 1-form η on M such that

$$g(X,\xi) = \eta(X), \ \ \phi^2 X = -X + \eta(X)\xi, \tag{1.7}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{1.8}$$

for any vector fields X, Y on M. In an almost contact metric manifold, we have

$$\phi\xi = 0, \ \ \eta \circ \phi = 0, \quad g(\phi X, Y) = -g(X, \phi Y). \tag{1.9}$$

Let $\overline{M}$ denote a n- dimensional kaehler manifold, i,e a smooth manifold with a (1,1) tensor field J and a Riemannian metric g such that J^2 = -I, $g(JX, JY) = g(X, Y)$, $\nabla J = 0$. Where X and Y are arbitrary vector fields on $\overline{M}$. I is the identity tensor field and $\bigtriangledown$ is the Riemannian connection of g. Now let M denote (n-1)- dimensional isometrically embedded orientable hypersurface of a Kaehler manifold $\overline{M}$. The metric g is same for both M and $\overline{M}$. If N denotes the unit normal vector field to M, then JN is tangent to M, then

$$JN = \xi, \quad JX = \phi X - \eta(X)N. \tag{1.10}$$

Let e_i $(i = 1, 2..., n-1)$ denote a local orthonormal basis of the tangent space to the contact hypersurface M of kaehler manifold $\overline{M}$. Then (e_i, N) is a local orthonormal basis of the tangent space to $\overline{M}$. We denote the arbitrary vector fields tangent to M by X, Y, Z and W. Further we denote the Ricci tensors of M, of type (0, 2) and (1, 1) by S and Q and the constant curvature of M by r corresponding objects of $\overline{M}$ are denoted by the same letters with overbars. In the Gauss equation

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(AX,Z)AY - g(AY,Z)AX. \tag{1.11}$$

Setting $X = W = e_i$, we get

$$\overline{S}(Y, Z,) = S(Y, Z) + g(AY, AZ) - hg(AY, Z). \tag{1.12}$$

$Y = Z = e_i$, we get

$$\overline{r} = r + (n-1) - h^2. \tag{1.13}$$

2. Curvature properties of genralised Sasakian space form and complex space form

A Riemannian manifold is said to be conformally flat if $R(X, Y).C = 0$, where C is the conformal curvature tensor is given by

$$\begin{aligned} \overline{C}(X, Y, Z, W) =& \overline{R}(X, Y, Z, W) - \frac{1}{n-2}[\overline{S}(Y, Z)g(X, W) - \overline{S}(X, Z)g(Y, W) + g(Y, Z)\overline{S}(X, W) \\ & - g(X, Z)\overline{S}(Y, W)] + \frac{\overline{r}}{(n-1)(n-2)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \end{aligned} \tag{2.1}$$

Using (1.11),(1.12) and (1.13) in (2.1), we get

$$\begin{aligned} \overline{C}(X, Y, Z, W) =& R(X, Y, Z, W) - \frac{1}{n-2}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W) + g(Y, Z)S(X, W) \\ & - g(X, Z)S(Y, W)] + \frac{r}{(n-1)(n-2)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ & - \frac{1}{n-2}[g(AY, AZ)g(X, W) - g(AX, AZ)g(Y, W) + g(AX, AW)g(Y, Z) - g(AY, AW) \\ & + h[-g(AY, Z)g(X, W) + g(AX, Z)g(Y, W) - g(AX, W)g(Y, Z) + g(AY, W)g(X, Z)] \\ & + \frac{(n-1) - h^2}{(n-1)(n-2)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \end{aligned} \tag{2.2}$$

from (2.2), we get

$$\begin{aligned}\overline{C}(X,Y,Z,W) =& C(X,Y,Z,W) - \frac{1}{n-2}\{g(AY,AZ)g(X,W) - g(AX,AZ)g(Y,W) \\ &+ g(AX,AW)g(Y,Z) - g(AY,AW)g(X,Z) + h[-g(AY,Z)g(X,W) \\ &+ g(AX,Z)g(Y,W) - g(AX,W)g(Y,Z) + g(AY,W)g(X,Z)]\} \\ &+ \frac{(n-1)-h^2}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)],\end{aligned} \tag{2.3}$$

From (1.2) and (2.1)

$$\begin{aligned}\overline{C}(X,Y,Z,W) =& \frac{-3c}{4(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ \frac{c}{4}[g(Z,JY)g(JX,W) - g(Z,JX)g(JX,W) + 2g(X,JY)g(JZ,W)],\end{aligned} \tag{2.4}$$

compare (2.4) and(2.3) we obtain

$$\begin{aligned}&R(X,Y,Z,W) - \frac{1}{n-2}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] + \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) \\ &- g(X,Z)g(Y,W)] = \frac{1}{n-2}\{g(AY,AZ)g(X,W) - g(AX,AZ)g(Y,W) \\ &+ g(AX,AW)g(Y,Z) - g(AY,AW)g(X,Z) + h[-g(AY,Z)g(X,W) \\ &+ g(AX,Z)g(Y,W) - g(AX,W)g(Y,Z) + g(AY,W)g(X,Z)]\} \\ &+ [\frac{h^2-(n-1)}{(n-1)(n-2)} - \frac{3c}{4(n-1)}][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ \frac{c}{4}[g(Z,JY)g(JX,W) - g(Z,JX)g(JX,W) + 2g(X,JY)g(JZ,W)],\end{aligned} \tag{2.5}$$

using (1.10) in (2.5)

$$\begin{aligned} R(X,Y,Z,W) =& [\frac{h^2-(n-1)}{(n-1)(n-2)} - \frac{3c}{4(n-1)} - \frac{r}{(n-1)(n-2)}][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+\frac{1}{n-2}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)S(X,W) \\ &-g(X,Z)S(Y,W] + \frac{1}{n-2}\{g(AY,AZ)g(X,W) - g(AX,AZ)g(Y,W) \\ &+g(AX,AW)g(Y,Z) - g(AY,AW)g(X,Z) + h[-g(AY,Z)g(X,W) \\ &+g(AX,Z)g(Y,W) - g(AX,W)g(Y,Z) + g(AY,W)g(X,Z)]\} \\ &+\frac{c}{4}[g(Z,\phi Y)g(\phi X,W) - g(Z,\phi X)g(\phi X,W) + 2g(X,\phi Y)g(\phi Z,W)], \end{aligned} \tag{2.6}$$

(2.6) will be reduced to real space form if

$$\begin{aligned} &\frac{1}{n-2}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)S(X,W) \\ &-g(X,Z)S(Y,W] + \frac{1}{n-2}\{g(AY,AZ)g(X,W) - g(AX,AZ)g(Y,W) \\ &+g(AX,AW)g(Y,Z) - g(AY,AW)g(X,Z) + h[-g(AY,Z)g(X,W) \\ &+g(AX,Z)g(Y,W) - g(AX,W)g(Y,Z) + g(AY,W)g(X,Z)]\} \\ &+\frac{c}{4}[g(Z,\phi Y)g(\phi X,W) - g(Z,\phi X)g(\phi X,W) + 2g(X,\phi Y)g(\phi Z,W)] = 0, \end{aligned} \tag{2.7}$$

$X = W = e_i$ in (2.7), we get

$$\begin{aligned} &\frac{n-3}{n-2}S(Y,Z) + [\frac{r+(n-1)-h^2}{(n-2)} - \frac{3c}{4}]g(Y,Z) + (n-3)g(AY,AZ) \\ &-h(n-3)g(AY,Z) - \frac{3c}{4}\eta(Y)\eta(Z) = 0, \end{aligned} \tag{2.8}$$

put $Y = Z = e_i$ in (2.8), we get

$$r = [\frac{(n^2-4n+5)h^2 - (2n^2-7n+7)}{(n-2)} - \frac{3c(n-2)^2}{4}]. \tag{2.9}$$

Taking (2.9) in (2.6), we get

$$\begin{aligned} R(X,Y,Z,W) =&P[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]\\ &+\frac{c}{4}[g(Z,\phi Y)g(\phi X,W)-g(Z,\phi X)g(\phi X,W)+2g(X,\phi Y)g(\phi Z,W)]\\ &+\frac{B}{n-2}[\eta(y)\eta(Z)g(X,W)-\eta(X)\eta(Z)g(Y,W)+\eta(X)\eta(W)g(Y,Z)\\ &-\eta(y)\eta(W)g(X,Z)]+\frac{1}{n-2}\{g(AY,AZ)g(X,W)-g(AX,AZ)g(Y,W)\\ &+g(AX,AW)g(Y,Z)-g(AY,AW)g(X,Z)+h[-g(AY,Z)g(X,W)\\ &+g(AX,Z)g(Y,W)-g(AX,W)g(Y,Z)+g(AY,W)g(X,Z)]\}, \end{aligned} \tag{2.10}$$

where $[P=\dfrac{-h^2(n^2-5n+7)+(n-1)(3n^2-4n+5)}{(n-1)(n-2)^2}+\dfrac{2A}{(n-2)}]$.
(2.10) reduces to generalized Sasakian space form if

$$\begin{aligned} &\frac{1}{n-2}\{g(AY,AZ)g(X,W)-g(AX,AZ)g(Y,W)+g(AX,AW)g(Y,Z)-g(AY,AW)g(X,Z)\\ &+h[-g(AY,Z)g(X,W)+g(AX,Z)g(Y,W)-g(AX,W)g(Y,Z)+g(AY,W)g(X,Z)]\}=0, \end{aligned} \tag{2.11}$$

$X=W=e_i$ in the above equation, we get

$$\frac{1}{n-2}[g(AY,AZ)(n-3)+h(n-3)g(AY,Z)+((n-1)-h^2)g(Y,Z)]=0,$$

put $Y=Z=e_i$ in the above equation, we get

$$(n-1)=h^2. \tag{2.12}$$

Using this in (2.10), we get

$$\begin{aligned} R(X,Y,Z,W) =&[\frac{2n^2+n-2)}{(n-2)^2}+\frac{2A}{(n-2)}][g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]\\ &+\frac{c}{4}[g(Z,\phi Y)g(\phi X,W)-g(Z,\phi X)g(\phi X,W)+2g(X,\phi Y)g(\phi Z,W)]\\ &+\frac{B}{n-2}[\eta(y)\eta(Z)g(X,W)-\eta(X)\eta(Z)g(Y,W)+\eta(X)\eta(W)g(Y,Z)\\ &-\eta(y)\eta(W)g(X,Z)] \end{aligned}$$

is generalized Sasakian space form if $f_1 = \dfrac{2n^2+n-2)}{(n-2)^2} + \dfrac{2A}{(n-2)}$, $f_2 = \dfrac{c}{4}$, $f_3 = \dfrac{-B}{n-2}$. Hence we state the following theorem.

Theorem 2.1. *An odd dimensional conformally flat generalised Sasakian space form is generalised Sasakian space form provided equation* (2.11) *holds.*

Let $M(F_1, F_2)$ be an odd- dimensional locally symmetric generalized complex space form. Then the curvature tensor is given by (1.3). From (1.3) we can see that

$$\overline{S}(Y,Z) = [(n-2)F_1 + 3F_2]g(Y,Z) + hg(AY,Z) - g(AY,AZ), \tag{2.13}$$

$$\overline{Q}Y = [(n-2)F_1 + 3F_2]Y + hAY - A^2Y, \tag{2.14}$$

$$\overline{r} = [(n-2)F_1 + 3F_2 - 1](n-1) + h^2. \tag{2.15}$$

From (1.3)

$$\overline{R}(X,Y)\xi = F_1[\eta(Y)X - \eta(X)Y] + \eta(AY)AX - \eta(AX)AY. \tag{2.16}$$

Hence from (2.16), we have

$$\overline{R}(X,\xi)\xi = F_1(-\phi^2 X) + \alpha\, A\, X - \alpha\, \eta(AX)\xi. \tag{2.17}$$

Using (2.16) and taking the covariant derivative of above equation with respect to Z. Since M is locally symmetric, we get

$$\begin{aligned} & - F_1\eta(X)\phi AZ + 3F_2\, g(X,AZ)N - 3F_2\eta(AZ)\eta(X)N + \eta(A\phi AZ)AX \\ & -\eta(AX)A\phi AZ - F_1\, g(X,\phi AZ)\xi - \alpha\, g(AX, A\phi AZ)\xi = -(ZF_1)\phi^2 X \\ & -F_1(\nabla_Z\phi)\phi X + \phi(\nabla_Z\phi)X) + \alpha(\nabla_Z A)X - \alpha((\nabla_Z\eta)AX)\xi \\ & -\alpha(\nabla_Z AX)\xi - \alpha\,\eta(AX)\nabla_Z\xi. \end{aligned} \tag{2.18}$$

$$(\nabla_Z\eta)Y = g(\nabla_Z\xi, Y) = g(\phi\, A\, Z, X), \tag{2.19}$$

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \tag{2.20}$$

using (2.19) and (2.20), we get

$$\begin{aligned}
&3F_2\, g(X, AZ)N - 3F_2\eta(AZ)\eta(X)N + \eta(A\phi AZ)AX \\
&-\eta(AX)A\phi AZ - F_1\, g(X, \phi AZ)\xi - \alpha\, g(AX, A\phi AZ)\xi = -(ZF_1)\phi^2 X \\
&+F_1\, g(AZ, \phi X)\xi + \alpha(\nabla_Z A)X - \alpha\, g(\phi AZ, AX)\xi \\
&-\alpha\eta((\nabla_Z A)X)\xi - \alpha\,\eta(AX)\phi AZ.
\end{aligned} \tag{2.21}$$

Contracting the above equation with respect to W, we get

$$\begin{aligned}
&\eta(A\phi AZ)g(AX, W) - \eta(AX)g(A\phi AZ, W) - F_1\, g(X, \phi AZ)\eta(W) - \alpha\, g(AX, A\phi AZ)\eta(W) \\
&= -\,(ZF_1)g(\phi^2 X, W) + F_1\, g(AZ, \phi X)\eta(W) + \alpha\, g((\nabla_Z A)X, W) - \alpha\, g(\phi AZ, AX)\eta(W) \\
&-\alpha\eta((\nabla_Z A)X)\eta(W) - \alpha\,\eta(AX)g(\phi AZ, W).
\end{aligned} \tag{2.22}$$

Setting $X = W = e_i$ in the above equation, we get

$$\eta(A\phi AZ)\,(h + \alpha^2) - \alpha\,\eta(A^2\phi AZ) - (ZF_1)(n-2) = 0. \tag{2.23}$$

Using the condition $A\,\phi = \phi\, A$, we get $ZF_1(n-2)$ =0.

$$F_1 \ is \ constant. \tag{2.24}$$

Contracting the (2.21) with respect to N, we get

$$3F_2 g(X, AZ) - 3F_2\,\eta(AZ)\eta(X) = 0 \tag{2.25}$$

Setting $X = Z = e_i$ in the above equation, we get $3F_2(h - \alpha) = 0$.

$$3F_2 = 0 \ for \ h \neq \alpha. \tag{2.26}$$

Using (2.24) and (2.26) in the equation (2.14) and (2.15), we get

$$\bar{Q}Y = [(n-2)K\,Y + h\,A\,Y - A^2 Y, \tag{2.27}$$

$$\bar{r} = [(n-2)F_1 - 1](n-1) + h^2. \tag{2.28}$$

Theorem 2.2. *An odd dimensional locally symmetric generalised complex space form has constant scalar curvature.*

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